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ON A NICE EMBEDDING AND THE ASCOLI THEOREM

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Abstract. Besides some remarks on hit–and–miss hyperspaces and (relative) compactness of unions of (relative) compact sets, and using these together with a natural map from a function space between topological spaces into a function space between their Vietoris hyperspaces, we derive Ascoli–like theorems for some set–open topologies, nearly related to the compact–open, and mainly for the compact-open topology itself. Some Ascoli–like statements are given, where all requirements are focused on the set of functions, whose (relative) compactness is in question, not on the base spaces.

1. Introduction

As general references to topological function spaces, the book of H. Poppe, [**28**], is recommended, as well as the book of R.A. McCoy and I. Ntantu, [**13**]. A really good overview, including historical aspects, is given by S. Naimpally in [**19**].

In the very interesting paper [**16**], Mizokami proved, that for Hausdorff topological spaces (X,τ), (Y,σ) the function space $C(X,Y)$, endowed with the compact-open topology τ_{co}, is isomorphic to a closed subspace of the function space $C\big(K(X),K(Y)\big)$, endowed with the pointwise topology τ_p, where the families $K(X)$, $K(Y)$ of compact sets are equipped with Vietoris topology. Inspired from this and the fundamental investigations of Poppe ([**22**], [**24**], [**28**]), we will try to find out here, what can be done, in order to derive Ascoli–like theorems, with the map, that Mizokami used.

We denote by $\mathfrak{F}(X)$, $\mathfrak{F}_0(X)$, $\mathfrak{F}(\varphi)$, $\mathfrak{F}_0(\varphi)$ the families of all filters on a set X, all ultrafilters on X, all refining filters of a filter φ and all refining ultrafilters of φ, respectively. A filter is not allowed to contain the empty set $\varnothing$. $\mathfrak{P}(X)$ denotes the power set of the set X, $\mathfrak{P}_0(X)$ means the power set without the empty set. By $\dot{x}$ we denote the singleton filter generated from the base $\big\{\{x\}\big\}$.

For a topological space (X,τ) we denote by q_τ the convergence on X induced by τ, i.e. the relation $q_\tau := \{(\varphi,x)\in\mathfrak{F}(X)\times X \mid \varphi\supseteq\dot{x}\cap\tau\}$ between the filters on X and the points of X.

Besides the absolutely basic facts concerning filters and ultrafilters, we will use here a fact, that is not commonly stated explicitly, although it seems to be sometimes explored, implicitly. So, we prove it here, for convenience.

Lemma 1.1 (Content Detector)**.** *Let X be a set, $\mathfrak{A}\subseteq\mathfrak{P}(X)$ and $\varphi\in\mathfrak{F}(X)$. Assume, $\mathfrak{A}$ is closed under finite unions of its elements. Then holds*

$$\varphi\cap\mathfrak{A}\neq\varnothing \iff \forall\psi\in\mathfrak{F}_0(\varphi):\psi\cap\mathfrak{A}\neq\varnothing,$$

i.e. a filter contains an $\mathfrak{A}$–set, iff each refining ultrafilter contains an $\mathfrak{A}$–set.

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Proof. Suppose $\forall\psi \in \mathfrak{F}_0(\varphi) : \exists A_\psi \in \mathfrak{A} : A_\psi \in \psi$. Now, assume $\varphi \cap \mathfrak{A} = \varnothing$. From this automatically follows $X \notin \mathfrak{A}$.

Consider $\mathfrak{B} := \{X \setminus A \mid A \in \mathfrak{A}\}$. Because of the closedness of $\mathfrak{A}$ under finite unions, $\mathfrak{B}$ is closed under finite intersection of its elements, and $\varnothing \notin \mathfrak{B}$, because $X \notin \mathfrak{A}$. For any $F \in \varphi$, $B \in \mathfrak{B}$ we have $F \cap B \neq \varnothing$, because $F \cap B = \varnothing$ would imply $F \subseteq X \setminus B \in \mathfrak{A}$ and therefore $\varphi \cap \mathfrak{A} \neq \varnothing$. So, $\varphi \cup \mathfrak{B}$ is a subbase of a filter and consequently, there exists an ultrafilter ψ, containing $\varphi \cup \mathfrak{B}$, therefore containing φ and the complement of every $\mathfrak{A}$–set — in contradiction to $\forall\psi \in \mathfrak{F}_0(\varphi) : \psi \cap \mathfrak{A} \neq \varnothing$.

The other direction of the lemma's statement is obvious. □

2. Some Remarks on Hit–and–Miss Hyperspaces

Let (X, τ) be a topological space. By $\mathrm{Cl}(X)$ and $K(X)$ we denote the family of all closed subsets and the set of all compact subsets of X, respectively. For $B \in \mathfrak{P}(X)$ and $\mathfrak{A} \subseteq \mathfrak{P}(X)$ we define $B^{-\mathfrak{A}} := \{A \in \mathfrak{A} \mid A \cap B \neq \varnothing\}$ (hit–set) and $B^{+\mathfrak{A}} := \{A \in \mathfrak{A} \mid A \cap B = \varnothing\}$ (miss–set). Please note, that this is a deviation from the otherwise and recently quite commonly used notation, where the same set would be denoted by "$(X \setminus B)^+$". By $\tau_{l,\mathfrak{A}}$ we denote the topology for $\mathfrak{A}$, generated by the subbase of all $G^{-\mathfrak{A}}$, $G \in \tau$. Now consider $\varnothing \neq \alpha \subseteq \mathfrak{P}(X)$; by $\tau_{\alpha,\mathfrak{A}}$ we denote the topology for $\mathfrak{A}$ which is generated from the subbase of all $B^{+\mathfrak{A}}$, $B \in \alpha$ and $G^{-\mathfrak{A}}$, $G \in \tau$. Of course, for every possible α we have $\tau_{l,\mathfrak{A}} \subseteq \tau_{\alpha,\mathfrak{A}}$; for $\alpha = \mathrm{Cl}(X)$ we get the Vietoris topology and for $\alpha = K(X)$ we get the Fell topology for $\mathfrak{A}$. If $\alpha = \Delta \subseteq \mathrm{Cl}(X)$, $\tau_{\alpha,\mathfrak{A}}$ is called Δ–topology [1] by Beer and Tamaki [**5**].

If X is a set, $\tau, \mathfrak{A}$ are subsets of $\mathfrak{P}(X)$, then we call $\mathfrak{A}$ *weakly complementary w.r.t.* τ, iff for every subset $\sigma \subseteq \tau$ there exists a subset $\mathfrak{B} \subseteq \mathfrak{A}$, s.t. $\bigcup_{B\in\mathfrak{B}} B = X \setminus \bigcup_{S\in\sigma} S$. Obviously, if τ is a topology on X, then $\mathrm{Cl}(X)$ and $K(X)$ are weakly complementary w.r.t. τ. Now, we ask for the reader's attention for a not even surprising, but quite useful set–theoretical lemma.

Lemma 2.1 (Covering Equivalence)**.** *Let X be a set, τ, $\mathfrak{A} \subseteq \mathfrak{P}(X)$ and $K \subseteq X$. Then holds*

$$\bigcup_{i\in I} G_i \supseteq K \implies \bigcup_{i\in I} G_i^{-\mathfrak{A}} \supseteq K^{-\mathfrak{A}}$$

for every collection G_i, $i \in I$, $G_i \in \tau$.

If $\mathfrak{A}$ is weakly complementary w.r.t. τ, then for every collection G_i, $i \in I$, $G_i \in \tau$ the implication

$$\bigcup_{i\in I} G_i \supseteq K \impliedby \bigcup_{i\in I} G_i^{-\mathfrak{A}} \supseteq K^{-\mathfrak{A}}$$

holds, too.

Corollary 2.2. *Let X be a set, $\tau, \mathfrak{A} \subseteq \mathfrak{P}(X)$ and $K \subseteq X$. Then holds*

$$\bigcup_{i\in I} G_i \supseteq K \iff \bigcup_{i\in I} G_i^{-\mathfrak{A}} \supseteq K^{-\mathfrak{A}}$$

for every collection $G_i, i \in I, G_i \in \tau$ if and only if $\mathfrak{A}$ is weakly complementary w.r.t. τ.

Lemma 2.3. *Let (X, τ) be a topological space and let $\mathfrak{A} \subseteq \mathfrak{P}(X)$ be weakly complementary w.r.t. τ. If $\mathfrak{A}_0 := \mathfrak{A} \setminus \{\varnothing\}$ is compact in $\tau_{l,\mathfrak{A}_0}$, then (X, τ) is compact.*

[1] Research on such topologies was initiated by H. Poppe in 1965, see [**22**], [**24**].

Definition 2.4. Let (X,τ) be a topological space. A subset $A \subseteq X$ is called *weak relative complete* in X, iff

$$\forall \varphi \in \mathfrak{F}(A) \cap q_\tau^{-1}(X) : \mathfrak{F}(\varphi) \cap q_\tau^{-1}(A) \neq \varnothing,$$

i.e. every filter φ on A, which converges in X, has a refinement, converging in A.

As another characterization, it follows immediately, that a subset A is weak relative complete in X, iff every ultrafilter on A, which converges in X, converges in A, too. Every closed and every compact subset is weak relative complete; for Hausdorff–spaces, weak relative completeness coincides with closedness. Every weak relative complete subset of a compact space is compact.

Theorem 2.5. *Let (X,τ) be a topological space, and let $\alpha \subseteq \mathfrak{P}(X)$ consist of weak relative complete subsets of X. Then holds for any $\mathfrak{A}$ with $\mathrm{Cl}(X) \subseteq \mathfrak{A} \subseteq \mathfrak{P}(X)$: $(\mathfrak{A}_0, \tau_{\alpha,\mathfrak{A}_0})$ is compact $\Longleftrightarrow$ (X,τ) is compact.*

Most of the well–known theorems for compactness w.r.t. the Fell — or the Vietoris–topology follow immediately from this. For some more explanation, especially for proofs of the foregoing statements, see [**3**].

Lemma 2.6. *Let (X,τ) be a topological space, $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ with $\mathrm{Cl}(X) \subseteq \mathfrak{A}$ and $\alpha \subseteq \mathrm{Cl}(X)$. If $R \subseteq X$ is relative compact in X, then $\mathfrak{P}_0(R) \cap \mathfrak{A}$ is relative compact*[2] *in $(\mathfrak{A}, \tau_\alpha)$.*

Proof. Let $\mathfrak{B} := \{O_i^{-\mathfrak{A}} \mid i \in I, O_i \in \tau\} \cup \{C_j^{+\mathfrak{A}} \mid j \in J, C_j \in \alpha\}$ be an open cover of $\mathfrak{A}$ by subbase elements of τ_α. Let $O := \bigcup_{i \in I} O_i$.

If $O = X$, then there exist finitely many $i_1, \ldots, i_n \in I$ with $\bigcup_{k=1}^n O_{i_k} \supseteq R$, because R is relative compact, and thus $\bigcup_{k=1}^n O_{i_k}^{-\mathfrak{A}} \supseteq R^{-\mathfrak{A}} \supseteq \mathfrak{P}_0(R) \cap \mathfrak{A}$, by Lemma 2.1.

If $O \neq X$, then $X \setminus O$ is nonempty and closed, but not contained in the $O_i^{-\mathfrak{A}}$–sets from $\mathfrak{B}$. Thus, there must exist a $j_0 \in J$ with $X \setminus O \in C_{j_0}^{+\mathfrak{A}}$, implying $C_{j_0} \subseteq O$. Now, we have $\mathfrak{P}_0(R) \cap \mathfrak{A} = (\mathfrak{P}_0(R) \cap C_{j_0}^{+\mathfrak{A}}) \cup (\mathfrak{P}_0(R) \cap C_{j_0}^{-\mathfrak{A}})$, and, of course, $\mathfrak{P}_0(R) \cap C_{j_0}^{+\mathfrak{A}}$ is covered just by $C_{j_0}^{+\mathfrak{A}} \in \mathfrak{B}$. So, we have to find a finite subcover for $(\mathfrak{P}_0(R) \cap C_{j_0}^{-\mathfrak{A}})$, if this is not empty. Observe, that $R \cap C_{j_0}$ is relative compact in X, because it is a subset of R. Furthermore, $\{O_i \mid i \in I\} \cup \{X \setminus C_{j_0}\}$ is an open cover of X. Thus we find again finitely many $i_1, \ldots, i_n \in I$, s.t. $\bigcup_{k=1}^n O_{i_k} \supseteq R \cap C_{j_0}$ (because $X \setminus C_{j_0}$ can be removed from any cover of $R \cap C_{j_0}$ without to lose the covering property). Therefore $\bigcup_{k=1}^n O_{i_k}^{-\mathfrak{A}} \supseteq (R \cap C_{j_0})^{-\mathfrak{A}}$, by Lemma 2.1. But $\mathfrak{P}_0(R) \cap C_{j_0}^{-\mathfrak{A}} \subseteq (R \cap C_{j_0})^{-\mathfrak{A}}$ holds, because any subset of R, which hits C_{j_0}, automatically hits $R \cap C_{j_0}$. $\square$

Proposition 2.7. *Let X be a set, $\mathfrak{X} \subseteq \mathfrak{P}(X)$ and $\mathfrak{M} \subseteq \mathfrak{X}$. Then holds*

$$\bigcup_{i \in I} C_i^{+\mathfrak{X}} \supseteq \mathfrak{M} \Longrightarrow \bigcup_{i \in I} C_i^c \supseteq \bigcup_{M \in \mathfrak{M}} M$$

for every collection C_i, $i \in I$.

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[2]According to [**2**], a subset R of a topological space (X,τ) is called relative compact in X, iff every ultrafilter on R converges in X, or equivalently, iff every open cover of X admits a finite subcover of R. Note, that this doesn't imply the compactness of the closure of R. See [**2**], [**28**].

Proof. For every $M \in \mathfrak{M}$ there must exist an $i_M \in I$ with $M \in C_{i_M}^{+x}$, because of $\bigcup_{i\in I} C_i^{+x} \supseteq \mathfrak{M}$. Thus $M \subseteq C_{i_M}^c \subseteq \bigcup_{i\in I} C_i^c$. □

Lemma 2.8. *Let (X,τ) be a topological space, let $\mathfrak{X}$ be the family of all relative compact subsets of X and let $\mathfrak{M} \subseteq \mathfrak{X}$ be relative compact in $\mathfrak{X}$ w.r.t. the upper Vietoris topology. Then*

$$R := \bigcup_{M\in\mathfrak{M}} M$$

is relative compact in (X,τ).

Proof. Let $\bigcup_{i\in I} O_i \supseteq X$ with $O_i \in \tau$, $i \in I$ an open cover of X. Because of the relative compactness of all $P \in \mathfrak{X}$, there is a finite subcover $O_{i_P^1}, \ldots, O_{i_P^{n_P}}$ for every $P \in \mathfrak{X}$, i.e. $O_P := \bigcup_{k=1}^{n_P} O_{i_P^k} \supseteq M$. Of course, $O_P \in \tau$ and so $(O_P)^c$ is closed w.r.t. τ. Furthermore, $P \cap O_P^c = \varnothing$, implying $P \in (O_P^c)^{+x}$. Thus we have $\mathfrak{X} \subseteq \bigcup_{P\in\mathfrak{X}} (O_P^c)^{+x}$, where the $(O_P^c)^{+x}$ are just open w.r.t. the upper–Vietoris topology. Because of the relative compactness of $\mathfrak{M}$ w.r.t. the upper–Vietoris topology, there must exist finitely many $P_1, \ldots, P_n \in \mathfrak{X}$ with $\mathfrak{M} \subseteq \bigcup_{j=1}^n (O_{P_j}^c)^{+x}$. Now, from Proposition 2.7 we get $R = \bigcup_{M\in\mathfrak{M}} M \subseteq \bigcup_{j=1}^n O_{P_j}$, where every O_{P_j} is a finite union of members of the original cover $\{O_i \mid i \in I\}$ by construction. □

Corollary 2.9. *Let (X,τ) be a topological space and let $\mathfrak{M} \subseteq \mathfrak{P}_0(X)$ consist of relative compact subsets of X. If $\mathfrak{M}$ is compact w.r.t. the upper–Vietoris topology, then*

$$R := \bigcup_{M\in\mathfrak{M}} M$$

is relative compact in (X,τ).

Proof. $\mathfrak{M}$ is compact and therefore relative compact in every set, which contains $\mathfrak{M}$, especially in the family of all relative compact subsets of X. So, Lemma 2.8 applies. □

For the Vietoris–topology on any $\mathfrak{A} \subseteq \mathfrak{P}_0(Z)$ for a topological space Z we will, from now on, use the base consisting of all sets

$$\langle O_1, \ldots, O_n\rangle_{\mathfrak{A}} := \mathfrak{A} \cap \left\{ M \in \mathfrak{P}_0(Z) \mid n \in \mathbb{N}, M \subseteq \bigcup_{i=1}^n O_i,\ \forall i : M \cap O_i \neq \varnothing \right\}$$

with open subsets O_i. If there seems to be no doubt, the index $\mathfrak{A}$ will be omitted from $\langle O_1, \ldots, O_n\rangle$.

3. Function Spaces

A very interesting and fairly wide class of function space structures, defined for Y^X or $C(X,Y)$, are the so called set–open topologies, examined in [**1**], [**28**]. According to [**28**], we use the following convention: Let X and Y be sets and $A \subseteq X$, $B \subseteq Y$; then let be $(A,B) := \{f \in Y^X \mid f(A) \subseteq B\}$. Now let X be a set, (Y,σ) a topological space and $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$. Then the topology $\tau_{\mathfrak{A}}$ on Y^X $\big($resp. $C(X,Y)\big)$, which is defined by the open subbase $\{(A,W) \mid A \in \mathfrak{A}, W \in \sigma\}$ is called the *set–open topology, generated by* $\mathfrak{A}$, or shortly the $\mathfrak{A}$*–open topology*.

Proposition 3.1. *Let X be a set, (Y, σ) a topological space and $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$, $\mathcal{F} \in \mathfrak{F}(Y^X)$, $f \in Y^X$. Then holds $(\mathcal{F}, f) \in q_{\tau_{\mathfrak{A}}}$ if and only if for all filters φ on X with a base consisting of $\mathfrak{A}$–members, holds $\mathcal{F}(\varphi) \supseteq f(\varphi) \cap \sigma$.*

Proof. Let $(\mathcal{F}, f) \in q_{\tau_{\mathfrak{A}}}$ and $\varphi \in \mathfrak{F}(X)$ with a base of $\mathfrak{A}$–members be given. Then for any $W \in \sigma \cap f(\varphi)$, there is an $A \in \mathfrak{A}$, such that $f(A) \subseteq W$. This means $f \in (A, W) \in \tau_{\mathfrak{A}}$, implying $(A, W) \in \mathcal{F}$ by $\mathcal{F} \xrightarrow{\tau_{\mathfrak{A}}} f$. So, we have $W \supseteq (A, W)(A) \in \mathcal{F}(\varphi)$.

If for all filters with a base in $\mathfrak{A}$ holds $\mathcal{F}(\varphi) \supseteq f(\varphi) \cap \sigma$, then we may chose the principal filters $[A]$ with $A \in \mathfrak{A}$ for φ to get $\mathcal{F}(A) \subseteq W$ for all $W \in \sigma \cap f(A)$, implying $(A, W) \in \mathcal{F}$ for any $A \in \mathfrak{A}, W \in \sigma$. □

Let (X, τ), (Y, σ) be topological spaces and $\mathfrak{A} \subseteq \mathfrak{P}(X)$. By $C_Y(\mathfrak{A})$ we denote the set $C_Y(\mathfrak{A}) := \{f(A) \mid A \in \mathfrak{A}, f \in C(X, Y)\}$ of all continuous images in Y of members of $\mathfrak{A}$.

We can naturally map the set Y^X, into the set $\mathfrak{P}(Y)^{\mathfrak{A}}$:

$$\mu : Y^X \to \mathfrak{P}(Y)^{\mathfrak{A}} : f \to \mu(f) : \forall A \in \mathfrak{A} : \mu(f)(A) = f(A)$$

Proposition 3.2. *Let (X, τ), (Y, σ) be topological spaces and $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$. If the function $f : X \to Y$ is continuous, then the function $\mu(f) : \mathfrak{A} \to \mathfrak{P}_0(Y)$ is continuous w.r.t. the Vietoris–topologies on $\mathfrak{A}$ and $\mathfrak{P}_0(Y)$.*

If $\mu(f)$ is continuous and $\mathfrak{A}$ is closed under finite unions and has the properties

(1) $\forall V \in \sigma, x \in f^{-1}(V) : \exists A_x \in \mathfrak{A} : x \in A_x \subseteq f^{-1}(V)$ *and*

(2) $\forall O \in \tau : \exists \mathfrak{B} \subseteq \mathfrak{A} : \bigcup_{B \in \mathfrak{B}} B = O$,

then f is continuous, too.

Proof. Let $\langle V_1, \ldots, V_n \rangle$ be an open base set of σ_V with all $V_i \in \sigma$. Then we have $A \in \mu(f)^{-1}(\langle V_1, \ldots, V_n \rangle) \Leftrightarrow A \in \mathfrak{A} \wedge f(A) \in \langle V_1, \ldots, V_n \rangle \Leftrightarrow A \in \mathfrak{A} \wedge f(A) \subseteq \bigcup_{i=1}^n V_i \wedge \forall i : f(A) \cap V_i \neq \varnothing \Leftrightarrow A \in \mathfrak{A} \wedge A \subseteq \bigcup_{i=1}^n f^{-1}(V_i) \wedge \forall i : A \cap f^{-1}(V_i) \neq \varnothing \Leftrightarrow A \in \langle f^{-1}(V_1), \ldots, f^{-1}(V_n) \rangle_{\mathfrak{A}}$. Thus $\mu(f)^{-1}(\langle V_1, \ldots, V_n \rangle) = \langle f^{-1}(V_1), \ldots, f^{-1}(V_n) \rangle$ is an open base set of τ_V on $\mathfrak{A}$, because all $f^{-1}(V_i)$ are open by the continuity of f.

Let $\mathfrak{A}$ have the mentioned properties, $\mu(f)$ be continuous and $V \in \sigma$. Then $(\mu(f))^{-1}(\langle V \rangle)$ is open in τ_V, i.e. $\forall A \in (\mu(f))^{-1}(\langle V \rangle) : \exists U_1(A), \ldots, U_{k(A)}(A) \in \tau : A \in \langle U_1(A), \ldots, U_{k(A)}(A) \rangle \subseteq (\mu(f))^{-1}(\langle V \rangle)$. Now, by (1) we find $\forall x \in f^{-1}(V) : \exists A_x \in \mathfrak{A} : x \in A_x \subseteq f^{-1}(V)$, implying $A_x \in (\mu(f))^{-1}(\langle V \rangle)$. Thus there are $U_1(A_x), \ldots, U_{k(A_x)}(A_x) \in \tau$ s.t. $A_x \in \langle U_1(A_x), \ldots, U_{k(A_x)}(A_x) \rangle_{\mathfrak{A}} \subseteq (\mu(f))^{-1}(\langle V \rangle)$, so by property (2) we get $\forall i = 1, \ldots, k(A_x) : \exists \mathfrak{B}_i \subseteq \mathfrak{A} : \bigcup_{B \in \mathfrak{B}_i} B = U_i(A_x)$ and then we take $\mathfrak{C} := \{\cup_{i=1}^{k(A_x)} B_i \mid \forall i : B_i \in \mathfrak{B}_i\}$ which is a subset of $\mathfrak{A}$ by closedness under finite unions. Now, we have $\bigcup_{C \in \mathfrak{C}} C = \bigcup_{i=1}^{k(A_x)} U_i(A_x)$, so obviously $\mathfrak{C} \subseteq \langle U_1(A_x), \ldots, U_{k(A_x)}(A_x) \rangle$, which is contained in $(\mu(f))^{-1}(\langle V \rangle)$, implying $\forall C \in \mathfrak{C} : \mu(f)(C) \subseteq V$ and therefore $f(\bigcup_{C \in \mathfrak{C}} C) = \bigcup_{C \in \mathfrak{C}} \mu(f)(C) \subseteq V$, implying $\bigcup_{C \in \mathfrak{C}} C \subseteq f^{-1}(V)$, so $O_x := \bigcup_{C \in \mathfrak{C}} C$ $(= \bigcup_{i=1}^{k(A_x)} U_i(A_x))$ is an open neighbourhood of x, contained in $f^{-1}(V)$. Taking these O_x for all $x \in f^{-1}(V)$, we find $f^{-1}(V)$ to be open. □

If $\mathfrak{A}$ is closed under finite unions and contains the singletons, then it has obviously all the properties required in the second part of the proposition. In any case, Proposition 3.2 ensures, that the image of $C(X,Y)$ under the mapping μ is a subset of $C\big(\mathfrak{A}, C_Y(\mathfrak{A})\big)$, where $\mathfrak{A}$ and $C_Y(\mathfrak{A})$ are equipped with Vietoris topology.

Proposition 3.3. *Let (X,τ), (Y,σ) be topological spaces, $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ and let $\mathcal{H} \subseteq Y^X$ be endowed with $\tau_{\mathfrak{A}}$. Then the map*

$$\mu : \mathcal{H} \to \mu(\mathcal{H}) := \{\mu(f) \mid \mu(f) : A \to f(A), f \in \mathcal{H}\} \subseteq \mathfrak{P}_0(Y)^{\mathfrak{A}}$$

is open, where $\mathfrak{A}$ and $\mathfrak{P}_0(Y)$ are equipped with Vietoris topology and $\mathfrak{P}_0(Y)^{\mathfrak{A}}$ with pointwise topology.

If $\mathcal{H} \subseteq C(X,Y)$ and $\mathfrak{A}$ has the property

$$\forall O \in \tau, A \in \mathfrak{A} : O \cap A \neq \varnothing \Rightarrow \exists A_O \in \mathfrak{A} : A_O \subseteq A \cap O, \tag{1}$$

then this map is continuous.

Proof. Let $\mathfrak{O} := \bigcap_{i=1}^n (A_i, O_i)$ with $A_i \in \mathfrak{A}$, $O_i \in \sigma$ be a basic open set of $\tau_{\mathfrak{A}}$. Then holds $f \in \mathfrak{O} \Leftrightarrow \forall i \in \{1,\dots,n\} : f(A_i) \subseteq O_i \Leftrightarrow \forall i \in \{1,\dots,n\} : \mu(f)(A_i) \in \langle O_i \rangle \Leftrightarrow \mu(f) \in \bigcap_{i=1}^n (\{A_i\}, \langle O_i \rangle)$, yielding $\mu(\mathfrak{O}) = \bigcap_{i=1}^n (\{A_i\}, \langle O_i \rangle)$, which is a basic open set of the pointwise topology on $\mu(\mathcal{H})$.

Let $(\mathcal{F}, f) \in q_{\tau_{\mathfrak{A}}}$, so by taking principal filters in Proposition 3.1, we get

$$\forall A \in \mathfrak{A} : \mathcal{F}(A) \supseteq [f(A)] \cap \sigma. \tag{2}$$

Now, let $A_0 \in \mathfrak{A}$ be given with $f(A_0) \in \langle V_1, \dots, V_n \rangle$ for some $V_1, \dots, V_n \in \sigma$. This means $f(A_0) \subseteq V_0 := \bigcup_{i=1}^n V_i$ and $\forall i \in \{1,\dots,n\} : f(A_0) \cap V_i \neq \varnothing$, implying $\forall i \in \{1,\dots,n\} : \exists A_i \in \mathfrak{A} : A_i \subseteq A_0 \cap f^{-1}(V_i)$, because of the required property of $\mathfrak{A}$ and the continuity of f. Then from (2) follows $\forall j \in \{0,1,\dots,n\} : \exists F_j \in \mathcal{F} : F_j(A_j) \subseteq V_j$, just meaning $\forall g \in F_j : g(A_j) \subseteq V_j$, thus from $A_j \subseteq A_0$ we get $\forall g \in F_j : g(A_0) \cap V_j \neq \varnothing$ and especially for $j = 0$ we have $F_0(A_0) \subseteq V_0$. But then $F := \bigcap_{j=0}^n F_j$ is an element of $\mathcal{F}$ and fulfills $\mu(F)(A_0) \subseteq \langle V_1, \dots, V_n \rangle$. This is valid for all basic open neighbourhoods of $f(A_0)$, so $\mu(\mathcal{F})(A_0)$ converges to $f(A_0) = \mu(f)(A_0)$ w.r.t. σ_V — for all $A_0 \in \mathfrak{A}$, thus $\mu(\mathcal{F})$ converges pointwise to $\mu(f)$. □

The property (1) is trivially fulfilled, if $\mathfrak{A}$ contains the singletons. Moreover, in this case we don't need to restrict the map to $C(X,Y)$, in order to prove its continuity.

Lemma 3.4. *Let (X,τ), (Y,σ) be topological spaces, let $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ contain the singletons and $\mathcal{H} \subseteq Y^X$ be endowed with $\tau_{\mathfrak{A}}$. Then the map*

$$\mu : \mathcal{H} \to \mu(\mathcal{H}) := \{\mu(f) \mid \mu(f) : A \to f(A), f \in \mathcal{H}\} \subseteq \mathfrak{P}_0(Y)^{\mathfrak{A}}$$

is open, continuous and bijective, where $\mathfrak{A}$ and $\mathfrak{P}_0(Y)$ are equipped with Vietoris topology, and $\mathfrak{P}_0(Y)^{\mathfrak{A}}$ with pointwise topology.

Proof. It's easy to see, that it is bijective, because each function f from X to Y is uniquely determined by the images of $\mu(f)$ on the singletons. Proposition 3.3 says, that it is open and, as is easy to see, the proof of continuity in Proposition 3.3 will work fine even without continuity of the $\tau_{\mathfrak{A}}$–limit function f of the filter $\mathcal{F}$, if we have in $\mathfrak{A}$ all singletons, because the combination of property (1) and continuity of f is only needed to ensure the existence of the subsets $\mathfrak{A} \ni A_i \subseteq A_0 \cap f^{-1}(V_i)$ for $i = 1, \dots, n$, but now we can always take singletons $\{x_i\}$ instead of these A_i. □

We will call this map

$$\mu : (Y^X, \tau_{\mathfrak{A}}) \to (\mu(Y^X), \tau_p) \subseteq (\mathfrak{P}_0(Y)^{\mathfrak{A}}, \tau_p) : f \to \mu(f) : A \to f(A)$$

the *Mizokami–map*, where $\mathfrak{A}$ and $\mathfrak{P}_0(Y)$ are endowed with Vietoris topology.

Proposition 3.5. *Let (X, τ), (Y, σ) be topological spaces and let $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ contain the singletons. If $\mathcal{F}$ is a filter on Y^X s.t. $\mu(\mathcal{F}) \xrightarrow{p} g \in \mathfrak{P}_0(Y)^{\mathfrak{A}}$, where $\mathfrak{P}_0(Y)$ is equipped with Vietoris topology, then there exists $g' \in Y^X$, with $\forall x \in X$: $g'(x) \in g(\{x\})$ and $\mathcal{F} \xrightarrow{p} g'$.*

Proof. $\mu(\mathcal{F}) \xrightarrow{p} g$ especially means for each singleton $\{x\} \subseteq X$, that $g(\{x\}) \in \langle V, Y\rangle$ with $V \in \sigma$ implies $\exists F \in \mathcal{F} : \forall f \in F : f(x) \in V$. Now, $g(\{x\})$ is never the empty set $\varnothing$, because this is not an element of our range space, so there exists a function $g' : X \to Y$ with $g'(x) \in g(\{x\})$ for all $x \in X$. But for arbitrary $y_x \in g(\{x\})$ and $V \in \dot{y}_x \cap \sigma$ we find $g(\{x\}) \in \langle V, Y\rangle$, and consequently $V \in \mathcal{F}(x)$. Thus $\mathcal{F}(x) \xrightarrow{\sigma} y_x$ and therefore $\mathcal{F}$ converges pointwise to g'. □

Definition 3.6. Let (X, τ), (Y, σ) be topological spaces and $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$. A subset $\mathcal{H} \subseteq Y^X$ is called $\mathfrak{A}$–*evenly continuous*, iff for all $A \in \mathfrak{A}$ holds

$$\forall \mathcal{F} \in \mathfrak{F}_0(\mathcal{H}), \varphi \in \mathfrak{F}(A), x \in X : (\mathcal{F}(x) \xrightarrow{\sigma} y) \wedge (\varphi \xrightarrow{\tau} x) \;\Rightarrow\; \mathcal{F}(\varphi) \xrightarrow{\sigma} y.$$

$\mathcal{H}$ is called *evenly continuous*, iff it is $\{X\}$–evenly continuous.

$\mathcal{H}$ is called *evenly continuous on a subset* K, iff the set of restricted functions $\mathcal{H}_{|K} := \{f_{|K} : K \to Y \mid f \in \mathcal{H}\}$ is evenly continuous.

Proposition 3.7. *Let (X, τ), (Y, σ) be topological spaces and $\mathcal{H} \subseteq C(X, Y)$.*

If $\mathcal{H}$ is $\{K\}$–evenly continuous for a subset $K \subseteq X$, then it is evenly continuous on K.

If Y Hausdorff, K a compact subset of X, and $\mathcal{H}$ evenly continuous on K, then it is $\{K\}$–evenly continuous.

Proof. The first statement follows trivially from the definition. So, let Y be Hausdorff, K compact and $\mathcal{H}$ be evenly continuous on K.

Furthermore, let $\mathcal{F}$ be a filter on $\mathcal{H}$, $x \in X$, $\varphi \in \mathfrak{F}(K)$ s.t. $\varphi \to x$ and $\mathcal{F}(x) \to y \in Y$. Now, we have for each refining ultrafilter φ_0 of φ, that it converges to x, too. But it must also converge to an element $a \in K$. Then for all continuous functions f follows $f(\varphi_0) \to f(a)$ and $f(\varphi_0) \to f(x)$, yielding $f(a) = f(x)$, because of the Hausdorffness of Y. Thus $\mathcal{F}(a) = \mathcal{F}(x)$, because all members of $\mathcal{F}$ consist of continuous functions. Consequently, $\mathcal{F}(a) \to y$, thus $\mathcal{F}(\varphi_0) \to y$, too, because $\mathcal{H}$ is evenly continuous on K. So, for an arbitrary $V \in \dot{y} \cap \sigma$ there must exist $F \in \mathcal{F}$, $P \in \varphi_0$, s.t. $F(P) \subseteq V$. Obviously, the family $\mathfrak{A}_V := \{A \subseteq X \mid \exists F \in \mathcal{F} : F(A) \subseteq V\}$ is closed under finite unions, because $\mathcal{F}$ is closed under finite intersections, and we have seen, that $\varphi_0 \cap \mathfrak{A}_V \neq \varnothing$ for every refining ultrafilter φ_0 of φ. So, Lemma 1.1 applies, yielding $\varphi \cap \mathfrak{A}_V \neq \varnothing$. This is valid for all open neighbourhoods of y, implying $\mathcal{F}(\varphi) \to y$. □

Proposition 3.8. *Let (X, τ), (Y, σ) be topological spaces, Y Hausdorff, and let $\mathcal{H}$ be a relative compact subset of $C(X, Y)$ w.r.t. the compact–open topology τ_{co}. Then $\mathcal{H}$ is evenly continuous on all compact subsets of X.*

Proof. Let $A \subseteq X$ be compact, $\varphi \in \mathfrak{F}(A)$, $a \in A$ and $\mathcal{F} \in \mathfrak{F}(\mathcal{H})$, s.t. $\mathcal{F}(a) \to y \in Y$ and $\varphi \to a$.

Then each refining ultrafilter $\mathcal{F}_0$ of $\mathcal{F}$ τ_{co}–converges to a continuous function g, because of the relative compactness of $\mathcal{H}$ in $C(X,Y)$. So, $y = g(a)$ follows, because $\mathcal{F}_0(a) \to y$, $\mathcal{F}_0$ converges especially pointwise to g and Y is Hausdorff. Moreover, $g(\varphi) \to y = g(a) \in g(A)$ holds, and $g(A)$ is compact and therefore closed, because A is compact, thus $g(A)$ is T_3, because Y is Hausdorff. Now, let $V_0 \in \dot{y} \cap \sigma$, then there exists $V_1 \in \sigma$, s.t. $y \in V_1 \cap g(A) \subseteq \overline{V_1 \cap g(A)} \subseteq V_0 \cap g(A)$. Furthermore, there exists $P_1 \in \varphi$, s.t. $g(P_1) \subseteq V_1 \cap g(A) \subseteq \overline{V_1 \cap g(A)}$ (remember, φ is a filter on A) and consequently $g^{-1}\big(\overline{V_1 \cap g(A)}\big) \in \varphi$ and $g^{-1}\big(\overline{V_1 \cap g(A)}\big)$ is closed in X, thus $B := g^{-1}\big(\overline{V_1 \cap g(A)}\big) \cap A$ is compact. But $g(B) \subseteq \overline{V_1 \cap g(A)} \subseteq V_0$ holds and $\mathcal{F}_0$ converges w.r.t. τ_{co} to g, thus $(B, V_0) \in \mathcal{F}_0$ and we have $B \in \varphi$, so $V_0 \in \mathcal{F}_0(\varphi)$ follows. Now, the family $\mathfrak{A}_{V_0} := \{F \subseteq \mathcal{H} \mid \exists P \in \varphi : F(P) \subseteq V_0\}$ is closed under finite unions of it's members, because φ is closed under finite intersections, and we have seen, that every refining ultrafilter of $\mathcal{F}$ contains a member of $\mathfrak{A}_{V_0}$. Thus, Lemma 1.1 applies, yielding $\mathcal{F} \cap \mathfrak{A}_{V_0} \neq \varnothing$, and this is valid for every $V_0 \in \dot{y} \cap \sigma$. So, $\mathcal{F}(\varphi)$ converges to y. □

Lemma 3.9. *Let (X,τ), (Y,σ) be topological spaces, $R \subseteq X$ a compact (resp. relative compact) subset and let $\mathcal{H} \subseteq C(X,Y)$ be $\{R\}$–evenly continuous. Then holds:*

If for every ultrafilter on R among its convergence-points exists a point $x \in R$ (resp. $x \in X$), s.t. the set $\mathcal{H}(x) := \{f(x) \mid f \in \mathcal{H}\}$ is compact (resp. relative compact) in Y, then $\mathcal{H}(R) := \{f(x) \mid f \in \mathcal{H}, x \in R\}$ is compact (relative compact) in Y, too.

Proof. Let $\psi \in \mathfrak{F}_0(\mathcal{H}(R))$. We have $\forall y \in \mathcal{H}(R) : \exists x_y \in R, f_y \in \mathcal{H} : y = f_y(x_y)$, thus there exists a map $\pi : \mathcal{H}(R) \to R \times \mathcal{H} : \pi(y) = (x_y, f_y), f_y(x_y) = y$. Now, $\pi(\psi)$ is an ultrafilter on $R \times \mathcal{H}$ and consequently $\pi_1\big(\pi(\psi)\big)$ and $\pi_2\big(\pi(\psi)\big)$ are ultrafilters on R and $\mathcal{H}$, respectively, where $\pi_1 : R \times \mathcal{H} \to R$ and $\pi_2 : R \times \mathcal{H} \to \mathcal{H}$ are the canonical projections. So, $\pi_1\big(\pi(\psi)\big)$ converges to a point $x_0 \in R$ (resp. $x_0 \in X$), s.t. $\mathcal{H}(x_0)$ is compact (resp. relative compact) in Y. Furthermore, $\pi_2\big(\pi(\psi)\big)(x_0)$ is an ultrafilter on $\mathcal{H}(x_0)$, thus it converges in $\mathcal{H}(x_0) \subseteq \mathcal{H}(R)$ (resp. in Y) to a point y_0. But then the $\{R\}$–even continuity of $\mathcal{H}$ implies that $\pi_2\big(\pi(\psi)\big)\,\big(\pi_1\big(\pi(\psi)\big)\big)$ converges to y_0, too. But we have naturally $\pi_2\big(\pi(\psi)\big)\,\big(\pi_1\big(\pi(\psi)\big)\big) \subseteq \psi$, so ψ converges in $\mathcal{H}(R)$ (resp. in Y). □

Lemma 3.10 (Essential Ascoli)**.** *Let (X,τ), (Y,σ) be topological spaces and $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$. Let $\mathcal{H} \subseteq C(X,Y)$ and $\mathcal{F}$ be an ultrafilter on $\mathcal{H}$, which converges pointwise to a function $g \in C(X,Y)$. Then the following hold:*

(1) *If $\mathfrak{A}$ consists only of relative compact subsets of X, $\mathcal{H}$ is $\mathfrak{A}$–evenly continuous and the images of all members of $\mathfrak{A}$ under g are closed in Y, then $\mu(\mathcal{F})$ converges pointwise to $\mu(g)$ in $C\big(\mathfrak{A}, C_Y(\mathfrak{A})\big)$.*

(2) *If $\mathfrak{A}$ consists only of compact subsets of X and $\mathcal{H}$ is evenly continuous on all members of $\mathfrak{A}$, then $\mu(\mathcal{F})$ converges pointwise to $\mu(g)$ in $C\big(\mathfrak{A}, C_Y(\mathfrak{A})\big)$.*

Proof. The continuity of $\mu(g)$ is ensured by Proposition 3.2.

Assume, $\mu(\mathcal{F})$ would not converge pointwise to $\mu(g)$. Then there are $A \in \mathfrak{A}$ and $V_1, \ldots, V_n \in \sigma$ such that $g(A) \in \langle V_1, \ldots, V_n \rangle$, but $\forall F \in \mathcal{F} : \exists f \in F : f(A) \notin \langle V_1, \ldots, V_n \rangle$. Thus, $\{f \in \mathcal{H} \mid f(A) \not\subseteq \bigcup_{i=1}^n V_i\} \cup \bigcup_{i=1}^n \{f \in \mathcal{H} \mid f(A) \cap V_i = \varnothing\}$ is a member of $\mathcal{F}$, because it's complement is not. Because $\mathcal{F}$ is an ultrafilter, one of the unified sets above must itself belong to $\mathcal{F}$.

Assume, it would $F_i := \{f \in \mathcal{H} \mid f(A) \cap V_i = \varnothing\} \in \mathcal{F}$ hold $(1 \leq i \leq n)$.

We have $g(A) \cap V_i \neq \varnothing$, implying $\exists x_g \in A : g(x_g) \in V_i$, so V_i is an open neighbourhood of $g(x_g)$. Thus $\exists F_g \in \mathcal{F} : \forall f \in F_g : f(x_g) \in V_i$, because of the pointwise convergence of $\mathcal{F}$ to g. But now $F_g \cap F_i = \varnothing$ holds — a contradiction to the filter–properties of $\mathcal{F}$.

So, $F_0 := \{f \in \mathcal{H} \mid f(A) \not\subseteq \bigcup_{i=1}^n V_i\} \in \mathcal{F}$ must hold. Let $V_A := \bigcup_{i=1}^n V_i$, then $\forall f \in F_0 : \exists x_f \in A : f(x_f) \notin V_A$. Thus, a map $\pi : F_0 \to A$ exists, s.t. $\forall f \in F_0 : f\bigl(\pi(f)\bigr) \notin V_A$. Then $\pi(\mathcal{F})$ is an ultrafilter on A, which must converge to a point $x_0 \in X$ (resp. $x_0 \in A$), because A is relative compact (resp. compact). Because of the pointwise convergence of $\mathcal{F}$ to g, it follows $\mathcal{F}(x_0) \xrightarrow{\sigma} g(x_0)$. From this and $\pi(\mathcal{F}) \xrightarrow{\tau} x_0$ follows $\mathcal{F}\bigl(\pi(\mathcal{F})\bigr) \xrightarrow{\sigma} g(x_0)$ by the $\mathfrak{A}$–even continuity of $\mathcal{H}$, just meaning

$$\forall V \in g(\dot{x}_0) \cap \sigma : \exists F_V \in \mathcal{F}, A_V \in \pi(\mathcal{F}) : F_V(A_V) \subseteq V. \tag{3}$$

On the other hand, $g\bigl(\pi(\mathcal{F})\bigr) \xrightarrow{\sigma} g(x_0)$ follows from the continuity of g. But $g\bigl(\pi(\mathcal{F})\bigr)$ is a filter on $g(A)$ and $g(A)$ is closed in the first of the lemma's statements, thus $g(x_0) \in g(A)$ holds, which follows in the second statement directly from $x_0 \in A$. Therefore, V_A is an open neighbourhood of $g(x_0)$ and from (3) we get $\exists F_V \in \mathcal{F}$, $A_V \in \pi(\mathcal{F}) : \forall f \in F_V,\ a \in A_V : f(a) \in V_A$. But then $F_V \cap \pi^{-1}(A_V) = \varnothing$ and $\pi^{-1}(A_V)$ is a member of $\mathcal{F}$ — a contradiction to the filter–properties of $\mathcal{F}$. So, our assumption $\mu(\mathcal{F}) \stackrel{p}{\not\to} \mu(g)$ must be false. □

Corollary 3.11. *Let (X, τ), (Y, σ) be topological spaces. Let $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ contain the singletons and consist only of relative compact subsets of X. Let $\mathcal{H} \subseteq C(X, Y)$ be $\mathfrak{A}$–evenly continuous and weakly relative complete in Y^X w.r.t. pointwise convergence and let all members of $\mathfrak{A}$ have closed images under elements of $\mathcal{H}$.*

Then $\mu(\mathcal{H})$ is weak relative complete in $\mathfrak{P}_0(Y)^{\mathfrak{A}}$ w.r.t. pointwise convergence, where $\mathfrak{P}_0(Y)$ is equipped with Vietoris topology.

Proof. Let $\mathcal{G}$ be an ultrafilter on $\mu(\mathcal{H})$, which converges pointwise to a function $g \in \mathfrak{P}_0(Y)^{\mathfrak{A}}$. At first, it is clear, that there exists an ultrafilter $\mathcal{F}$ on $\mathcal{H}$, s.t. $\mathcal{G} = \mu(\mathcal{F})$. From g we derive a function $g' : X \to Y$: for all singletons $\{x\} \in \mathfrak{A}$, we can chose an element y_x from $g(\{x\})$, because the empty set doesn't belong to our range space. Then for each open neighbourhood V of y_x we find $g(\{x\}) \in \langle V, Y \rangle$, so there must exist a $F \in \mathcal{F}$ with $\forall f \in F : \mu(f)(\{x\}) \in \langle V, Y \rangle$, just implying $\mathcal{F} \xrightarrow{p} g'$, where g' is chosen s.t. $g' : X \to Y : g'(x) := y_x \in g(\{x\})$. Now, because of the weak relative completeness of $\mathcal{H}$, there must exist a function $g_1 \in \mathcal{H}$ with $\mathcal{F} \xrightarrow{p} g_1$. From Lemma 3.10 follows $\mu(\mathcal{F}) = \mathcal{G} \xrightarrow{p} \mu(g_1) \in \mu(\mathcal{H})$. □

Corollary 3.12. *Let (X, τ), (Y, σ) be topological spaces, Y Hausdorff. Let $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ contain the singletons and consist only of compact subsets of X. Let $\mathcal{H} \subseteq C(X, Y)$ be $\mathfrak{A}$–evenly continuous and weakly relative complete in Y^X w.r.t. pointwise convergence.*

Then $\mu(\mathcal{H})$ is closed in $K(Y)^{\mathfrak{A}}$ w.r.t. pointwise convergence, where $K(Y)$ is equipped with Vietoris topology.

Proof. Of course, compact subsets are relative compact. Continuous images of compact sets are compact and therefore closed in the Hausdorff–space Y. So, Corollary 3.11 applies, yielding $\mu(\mathcal{H})$ to be weak relative complete in $\mathfrak{P}_0(Y)^{\mathfrak{A}}$ and consequently in $K(Y)^{\mathfrak{A}}$ $\big($since $K(Y)^{\mathfrak{A}}$ is a subspace of $\mathfrak{P}_0(Y)^{\mathfrak{A}}$ w.r.t. pointwise convergence$\big)$. But if Y is Hausdorff, then $K(Y)$ with Vietoris–topology is, and consequently, the function space is Hausdorff, too. So, as a weak relative complete subset, $\mu(\mathcal{H})$ is closed. □

Note, that this is somewhat other than Mizokami showed. We require the additional condition of $\mathfrak{A}$–even continuity and get the stronger result of closedness in $K(Y)^{\mathfrak{A}}$, not only in $C\big(\mathfrak{A}, C_Y(\mathfrak{A})\big)$.

Corollary 3.13. *Let $(X,\tau),(Y,\sigma)$ be topological spaces, Y Hausdorff and T_3. Let $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ contain the singletons and consist only of compact subsets of X. Let $\mathcal{H} \subseteq C(X,Y)$ be evenly continuous and weakly relative complete in $C(X,Y)$ w.r.t. pointwise convergence. Then $\mu(\mathcal{H})$ is closed in $K(Y)^{\mathfrak{A}}$.*

Proof. If an ultrafilter $\mathcal{F}$ on $\mathcal{H}$ converges pointwise in Y^X to a function g, then from the even continuity of $\mathcal{H}$ follows, that $\mathcal{F}$ converges continuously to g and then with theorem 30 in [**2**] from T_3 the continuity of g. So, $\mathcal{F}$ converges in $C(X,Y)$ and therefore in $\mathcal{H}$, because of the weak relative completeness in $C(X,Y)$. Thus, $\mathcal{H}$ is indeed weak relative complete in Y^X and Corollary 3.12 applies. □

Theorem 3.14. *Let (X,τ), (Y,σ) be topological spaces and let $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ contain the singletons. Then a set $\mathcal{H} \subseteq Y^X$ is relative compact in $(Y^X, \tau_{\mathfrak{A}})$ if and only if*

(1) *For all ultrafilters $\mathcal{F}$ on $\mathcal{H}$ with $\mathcal{F} \xrightarrow{p} f \in Y^X$ exists a function $g \in Y^X$, s.t. $\mu(\mathcal{F}) \xrightarrow{p} \mu(g) \in \mathfrak{P}_0(Y)^{\mathfrak{A}}$, where $\mathfrak{P}_0(Y)$ is equipped with Vietoris topology, and*

(2) *for all $A \in \mathfrak{A}$, the family $\mu(\mathcal{H})(A) := \{f(A) \mid f \in \mathcal{H}\}$ is relative compact in $\mathfrak{P}_0(Y)$ w.r.t. Vietoris topology.*

Proof. Because $\mathfrak{A}$ contains the singletons, the Mizokami–map $\mu : (\mathcal{H}, \tau_{\mathfrak{A}}) \to (\mu(\mathcal{H}), \tau_p)$ is continuous, open and bijective by Lemma 3.4. Now, $(\mathfrak{P}_0(Y)^{\mathfrak{A}}, \tau_p)$ is naturally isomorphic to $\prod_{A\in\mathfrak{A}} \mathfrak{P}_0(Y)_A$ with Tychonoff product topology, where all $\mathfrak{P}_0(Y)_A$ are clones of $\mathfrak{P}_0(Y)$ (see [**28**], 2.2), let

$$\pi : (\mathfrak{P}_0(Y)^{\mathfrak{A}}, \tau_p) \to \prod_{A\in\mathfrak{A}} \mathfrak{P}_0(Y)_A : f \to \big(f(A)\big)_{A\in\mathfrak{A}}$$

be the isomorphism. Then in fact, $\pi(\mu(\mathcal{H}))$ is just a subset of the product $\prod_{A\in\mathfrak{A}} \mu(\mathcal{H})(A)$.

Let (1) and (2) be fulfilled. Then all $\mu(\mathcal{H})(A)$ are relative compact in $\mathfrak{P}_0(Y)$ by (2), so the product $\prod_{A\in\mathfrak{A}} \mu(\mathcal{H})(A)$ is relative compact in $\prod_{A\in\mathfrak{A}} \mathfrak{P}_0(Y)_A$ by the Tychonoff–theorem for relative compact subsets (see 1.44 in [**28**]). Thus, as a subset of a relative compact set, $\pi(\mu(\mathcal{H}))$ itself is relative compact in $\prod_{A\in\mathfrak{A}} \mathfrak{P}_0(Y)_A$. Let $\mathcal{F}$ be an ultrafilter on $\mathcal{H}$, then $\pi(\mu(\mathcal{F}))$ is an ultrafilter on $\pi(\mu(\mathcal{H}))$, which now must converge in $\prod_{A\in\mathfrak{A}} \mathfrak{P}_0(Y)_A$, implying $\mu(\mathcal{F})$ converges pointwise to a function

$f \in \mathfrak{P}_0(Y)^{\mathfrak{A}}$, by isomorphism. Then by Proposition 3.5, $\mathcal{F}$ converges pointwise to a function $f' \in Y^X$. From (1) now follows the existence of a function $g \in Y^X$ with $\mu(\mathcal{F}) \stackrel{p}{\to} \mu(g)$ and thus $\mathcal{F} \stackrel{\tau_{\mathfrak{A}}}{\to} g$, because the Mizokami–map is open between $(Y^X, \tau_{\mathfrak{A}})$ and $(\mu(Y^X), \tau_p)$, by Lemma 3.4.

If otherwise $\mathcal{H}$ is relative compact in Y^X w.r.t. $\tau_{\mathfrak{A}}$, then every ultrafilter $\mathcal{F}$ on $\mathcal{H}$ $\tau_{\mathfrak{A}}$–converges to a function $g \in Y^X$, and therefore $\mu(\mathcal{F})$ converges pointwise to $\mu(g)$ by the continuity of the Mizokami–map, and of course, $\mathcal{F}$ converges pointwise to g, because $\mathfrak{A}$ contains the singletons — so, (1) is fulfilled. Furthermore, an ultrafilter $\mathcal{G}$ on $\mu(\mathcal{H})(A)$ induces an ultrafilter $\mathcal{G}'$ on $\mu(\mathcal{H})$, whose evaluation on A is just $\mathcal{G}$, and therefore an ultrafilter $\mathcal{F}$ on $\mathcal{H}$ exists, with $\mu(\mathcal{F}) = \mathcal{G}'$, by bijectivity of the Mizokami–map. Now, $\mathcal{F}$ $\tau_{\mathfrak{A}}$–converges to a function $f \in Y^X$, by the relative compactness of $\mathcal{H}$, thus $\mu(\mathcal{F})(A) = \mathcal{G}$ converges to $\mu(f)(A)$, because of the continuity of the Mizokami–map — so, (2) is fulfilled. □

Corollary 3.15. *Let (X, τ), (Y, σ) be topological spaces and let $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ contain the singletons. Then a set $\mathcal{H} \subseteq Y^X$ is relative compact in $(Y^X, \tau_{\mathfrak{A}})$, if*

(1) *For all ultrafilters $\mathcal{F}$ on $\mathcal{H}$ with $\mathcal{F} \stackrel{p}{\to} f \in Y^X$ exists a function $g \in Y^X$, s.t. $\mu(\mathcal{F}) \stackrel{p}{\to} \mu(g) \in \mathfrak{P}_0(Y)^{\mathfrak{A}}$, where $\mathfrak{P}_0(Y)$ is equipped with Vietoris topology, and*

(2) *for all $A \in \mathfrak{A}$, the set $\mathcal{H}(A) := \bigcup_{f \in \mathcal{H}} f(A)$ is relative compact in Y.*

Proof. If $\mathcal{H}(A)$ is relative compact in Y, then $\mathfrak{P}_0(\mathcal{H}(A))$ is relative compact in $\mathfrak{P}_0(Y)$ w.r.t. Vietoris topology, by Lemma 2.6, thus the subset $\mu(\mathcal{H})(A)$ is, and then the Theorem 3.14 applies. □

Corollary 3.16. *Let (X, τ), (Y, σ) be topological spaces and let $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ consist only of relative compact subsets of X and contain the singletons. Let $\mathcal{H} \subseteq C(X, Y)$ have the following properties:*

(1) *$\mathcal{H}$ is weakly relative complete in Y^X w.r.t. pointwise convergence,*

(2) *$\mathcal{H}$ is $\mathfrak{A}$–evenly continuous,*

(3) *the images of all members of $\mathfrak{A}$ under elements of $\mathcal{H}$ are closed in Y and*

(4) *for all $A \in \mathfrak{A}$, each ultrafilter φ on A converges to a point $x_0 \in X$, s.t. $\mathcal{H}(x_0) := \{f(x_0) \mid f \in \mathcal{H}\}$ is relative compact in Y.*

Then $\mathcal{H}$ is compact w.r.t. $\tau_{\mathfrak{A}}$.

If otherwise $\mathcal{H}$ is compact w.r.t. $\tau_{\mathfrak{A}}$, then (1) follows and for all $A \in \mathfrak{A}$ is $\mathcal{H}(A) := \bigcup_{f \in \mathcal{H}} f(A)$ relative compact in Y.

Proof. Condition (1) ensures, that every ultrafilter $\mathcal{F}$ on $\mathcal{H}$, which pointwise converges in Y^X, converges in $\mathcal{H}$, too. From (2) and (3) follows, that for each ultrafilter $\mathcal{F}$ on $\mathcal{H}$ always $\mathcal{F} \stackrel{p}{\to} g \in C(X, Y)$ implies $\mu(\mathcal{F}) \stackrel{p}{\to} \mu(g)$, by Lemma 3.10(1). From (2) and (4) follows the relative compactness of all $\mathcal{H}(A)$ for $A \in \mathfrak{A}$, by Lemma 3.9.

Thus, Corollary 3.15 applies, yielding the relative compactness of $\mathcal{H}$ in Y^X. Now, from (1), which is carried over to $\tau_{\mathfrak{A}}$, by Corollary 3.11, follows the compactness.

If otherwise $\mathcal{H}$ is compact w.r.t. $\tau_{\mathfrak{A}}$, then it is compact w.r.t. pointwise convergence, too, and so (1) follows trivially, and the relative compactness of all $\mathcal{H}(A)$, $A \in \mathfrak{A}$ follows by corollary 2.9, because $\mu(\mathcal{H})(A)$ is compact w.r.t. the

Vietoris topology by the continuity of both, the Mizolami–map and the projections $p_A : \mathfrak{P}_0(Y)^{\mathfrak{A}} \to \mathfrak{P}_0(Y) : g \to g(A)$. □

Corollary 3.17. *Let (X,τ), (Y,σ) be topological spaces and let $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ consist only of compact subsets of X and contain the singletons. Let $\mathcal{H} \subseteq C(X,Y)$ have the following properties:*

(1) *$\mathcal{H}$ is weak relative complete in Y^X w.r.t. pointwise convergence,*

(2) *$\mathcal{H}$ is $\mathfrak{A}$–evenly continuous,*

(3) *for all $A \in \mathfrak{A}$, each ultrafilter φ on A converges to a point $x_0 \in X$, s.t. $\mathcal{H}(x_0) := \{f(x_0) \mid f \in \mathcal{H}\}$ is relative compact in Y.*

Then $\mathcal{H}$ is compact w.r.t. $\tau_{\mathfrak{A}}$.

If otherwise $\mathcal{H}$ is compact w.r.t. $\tau_{\mathfrak{A}}$, then (1) *follows and for all $A \in \mathfrak{A}$ is $\mathcal{H}(A) := \bigcup_{f\in\mathcal{H}} f(A)$ compact in Y.*

Proof. Copy the proof of Corollary 3.16, but use part (2) of Lemma 3.10, instead of part (1), then the closedness of the images is not needed. □

Note, that all requirements, in order to make $\mathcal{H}$ compact, are focused on $\mathcal{H}$ and $\mathfrak{A}$. There is no condition concerning the spaces X, Y (except, that they should be topological spaces). This seems to be natural, because in fact, the compactness of $\mathcal{H}$ is in question, not the compactness of X or Y. But, of course, special properties of the range space may simplify the requirements, as the following shows. At this point, we come back to a classical looking form of Ascoli–like statements.

Corollary 3.18. *Let (X,τ), (Y,σ) be topological spaces, Y Hausdorff. Then a set of functions $\mathcal{H} \subseteq C(X,Y)$ is compact w.r.t. the compact–open topology τ_{co}, if and only if it has the following properties:*

(1) *$\mathcal{H}$ is closed in Y^X w.r.t. pointwise convergence,*

(2) *$\mathcal{H}$ is evenly continuous on all compact subsets and*

(3) *for all $x \in X$ is $\mathcal{H}(x) := \{f(x) \mid f \in \mathcal{H}\}$ relative compact in Y.*

Proof. Proposition 3.7 ensures, that we can apply Lemma 3.9, yielding

(4) For all $A \in K(X)$ is $\mathcal{H}(A) := \bigcup_{f\in\mathcal{H}} f(A)$ compact in Y.

Now, let $\mathfrak{A} := K_0(X)$, the set of all nonempty compact subsets of X, so $\tau_{\mathfrak{A}}$ is just the compact–open topology τ_{co}. Because Y is Hausdorff, from (2) we get the $\mathfrak{A}$–even continuity of $\mathcal{H}$, by Proposition 3.7, so, if (1), (2), (4) are fulfilled, Corollary 3.17 applies, yielding $\mathcal{H}$ to be compact w.r.t. τ_{co}.

If otherwise $\mathcal{H}$ is compact w.r.t. τ_{co}, we get (1) and (4) (thus (3)) from Corollary 3.17 again, and (2) from Proposition 3.8. □

Note, that sufficiency of the three conditions for compactness w.r.t. τ_{co} could be also obtained from Theorem 3.36(I) in [**28**], together with Proposition 3.7 here.

To require closedness for $\mathcal{H}$ here, instead of weak relative completeness as in Corollary 3.17, is not really stronger, because Y^X is Hausdorff w.r.t. pointwise convergence, whenever Y is, and so closedness and weak relative completeness coincide.

This corollary is just a repaired version of Edwards' statement 3.13 in [**9**], where only closedness of $\mathcal{H}$ in $C(X,Y)$ — not in Y^X — is required, 3.18(4) is required instead of 3.18(3) and condition 3.18(2) is omitted[3]. The following shows, that this would indeed not be enough to get compactness for $\mathcal{H}$, and especially, that an essential reason for this is the absence of properties like some kind of even continuity, for example.

Example 3.19. Let $X = Y = [0,1] \subseteq \mathbb{R}$ be equipped with euclidian topology. Now, let

$$c_s : X \to Y : c_s(x) = s, \qquad s \in [0,1]$$

and let $\mathcal{H}_1 := \{c_s \mid 0 \leq s \leq 1\}$. Furthermore, let

$$w_n : X \to Y : w_n(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{3n} \\ 3nx - 1 & \frac{1}{3n} < x \leq \frac{2}{3n} \\ -3nx + 3 & \frac{2}{3n} < x \leq \frac{1}{n} \\ 0 & \frac{1}{n} < x \leq 1 \end{cases}$$

with $n \in \mathbb{N}, n \geq 2$ and then let $\mathcal{H}_2 := \{w_n \mid n \in \mathbb{N}, n \geq 2\}$.

Then $\mathcal{H} := \mathcal{H}_1 \cup \mathcal{H}_2$ is closed in Y^X w.r.t. pointwise convergence and for all subsets K (especially for all compact subsets) of X is $\mathcal{H}(K)$ compact. But $\mathcal{H}$ is not compact w.r.t. the compact–open topology.

Proof. It is clear, that $\mathcal{H}_1(K) = [0,1]$ for all nonempty subsets$K$ of X. So, in any case $\mathcal{H}_2(K) \subseteq \mathcal{H}_1(K)$ and consequently $\mathcal{H}(K) = \mathcal{H}_1(K) \cup \mathcal{H}_2(K) = \mathcal{H}_1(K)$ is compact.

To see, that $\mathcal{H}$ is closed in Y^X, let $\mathcal{F}$ be an ultrafilter on $\mathcal{H}$, which converges pointwise to a function $g \in Y^X$. Then $\mathcal{F}$ either contains $\mathcal{H}_1$ or $\mathcal{H}_2$, because it is an ultrafilter. If $\mathcal{F}$ contains $\mathcal{H}_1$, then its evaluation filter on every point of X is the same - and as an ultrafilter in the compact Y this converges to a point of Y, thus $\mathcal{F}$ converges pointwise to the associated constant function. If $\mathcal{F}$ contains $\mathcal{H}_2$, then either $\mathcal{F}$ is a singleton–filter (and therefore converges pointwise to its generating element of $\mathcal{H}_2$) or it contains the filter $\mathcal{G} := \big[\big\{\{w_k \mid k \geq n\} \mid n \in \mathbb{N}, n \geq 2\big\}\big]$. But this filter obviously converges pointwise to $c_0 \in \mathcal{H}$, and so all refining ultrafilters do. Thus, $\mathcal{F}$ converges in $\mathcal{H}$, whenever it converges in Y^X, so $\mathcal{H}$ is closed in Y^X w.r.t. pointwise convergence, because Y^X is Hausdorff.

Otherwise, just the filter $\mathcal{G}$ fails to converge w.r.t. the compact-open topology τ_{co}: the convergence w.r.t. τ_{co} coincides with continuous convergence, because X is locally compact. The only function, to which $\mathcal{G}$ could converge w.r.t. τ_{co} is c_0, because it converges pointwise only to this function. So, for the neighbourhood–filter $\boldsymbol{U}(0)$ of zero, $\mathcal{G}(\boldsymbol{U}(0))$ should converge to 0 — but it doesn't, because for any $G \in \mathcal{G}$ and any open neighbourhood U of 0 we find $1 \in G(U)$. Thus, there

[3] His statement is false. Nevertheless, the gap in his argumentation is quite sophisticated: he deals with functions from a hyperspace to a hyperspace and forgets at an essential position, that not all of these functions need to be induced from a function between the base spaces, especially, they are not enforced to fulfill $f(A) = \bigcup_{x \in A} f(\{x\})$.

must exist a refining ultrafilter of $\mathcal{G}$, which doesn't τ_{co}–converge to c_0 and therefore completely fails to converge w.r.t. τ_{co}. □

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