THE TRACE FUNCTION EXPANSION FOR SPHERICAL POLYGONS

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Abstract. The full asymptotic expansion of the trace of the heat semi–group,
\[ \text{tr}(e^{-\Delta \Omega t}), \]
where \(-\Delta \Omega\) is the Dirichlet Laplace-Beltrami operator acting on \(L^2(\Omega)\) for geodesic spherical polygons \(\Omega \subset S^2\), is derived in half–powers of \(t\), and the coefficients determined explicitly.

Let \(\Omega\) be a non-empty open connected set in \(S^2 = \{ x \in \mathbb{R}^3 : |x| = 1 \}\) with a piecewise geodesic boundary \(\partial \Omega\) and interior angles \(\gamma_1, \ldots, \gamma_M\). We refer to \(\Omega\) as a geodesic spherical polygon. In this paper we study the asymptotic behaviour as \(t \downarrow 0\) of the trace function,
\[ Z(\Omega, t) := \text{tr}(e^{-\Delta \Omega t}), \]
where \(-\Delta \Omega\) is the Dirichlet Laplace–Beltrami operator acting on \(L^2(\Omega)\).

As well as being of interest to mathematicians, the asymptotic expansion of the heat kernel and the trace function has applications in theoretical physics. The asymptotic expansion gives a good approximation of curved space-time propagators, and in computing the Casimir energy. We refer to [3] for an extensive review of the applications of spectral geometry in theoretical physics.

In [5] Chang and Dowker computed the vacuum energy on orbifold factors of spheres, where they utilised the trace expansion on spherical triangles and quadrilaterals of the 2–sphere. The asymptotic expansion that we give in this paper for geodesic spherical polygons has direct relevance to their calculations and applications. Furthermore, the trace expansion identifies the poles of the associated spectral zeta function, which is used in the calculations of functional determinants, which are of interest to both mathematicians [17] and physicists [7].

The results in Theorem 2 and Corollary 3 extends those of Kac [12] and van den Berg and Srisatkunarajah [2] from a polygon in \(\mathbb{R}^2\) to a spherical polygon in \(S^2\).

Let \(\{\lambda_j, \varphi_j\}\) be the spectral resolution of \(-\Delta \Omega\). Then it is well–known that the Dirichlet eigenvalues form an increasing sequence, \(0 < \lambda_0 \leq \lambda_1 \leq \cdots\), accumulating at infinity. Moreover, the heat semi–group \(e^{-\Delta \Omega t}\) on \(L^2(\Omega)\) has a strictly positive \(C^\infty\) heat kernel on \(\Omega \times \Omega \times (0, \infty)\), see for example Chavel [6], which we denote by \(K_\Omega(\theta_1, \theta_2, t)\). Provided \(|\Omega| < \infty\) it is well–known that,
\[ K_\Omega(\theta_1, \theta_2, t) = \sum_{j=0}^{\infty} e^{-\lambda_j t} \varphi_j(\theta_1) \varphi_j(\theta_2), \]
which converges absolutely on compact subsets of \(\Omega \times \Omega \times (0, \infty)\).

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Carleman [4], introduced the trace function \( Z(\Omega, \cdot) : (0, \infty) \to (0, \infty) \) given by,
\[
Z(\Omega, t) = \text{tr}(e^{-\Delta_\Omega t})
\]
and the representation (2), together with Fubini’s theorem, gives
\[
Z(\Omega, t) = \sum_{j=0}^{\infty} e^{-\lambda_j t}.
\]

The trace function (sometimes referred to as the partition function) is of great importance to the spectral geometer, who is interested in obtaining global geometric information from the spectrum of the operator. It is well-known when \( \Omega \) is a compact \( m \)-dimensional Riemannian manifold with a smooth boundary \( \partial \Omega \), see for example Greiner [9] and Seeley [19], that as \( t \downarrow 0 \) and for any \( N \in \mathbb{N} \),
\[
Z(\Omega, t) = \sum_{n=0}^{N} a_n(\Omega) t^{(n-m)/2} + O(t^{(N+1-m)/2}),
\]
with
\[
a_0(\Omega) = \frac{|\Omega|}{(4\pi)^{m/2}}, \quad a_1(\Omega) = -\frac{1}{4} \frac{|\partial \Omega|}{(4\pi)^{(m-1)/2}}.
\]
The coefficients \( a_0(\Omega), a_1(\Omega) \) and \( a_2(\Omega) \) were computed by McKean and Singer in [14].

In [12] Kac obtained for a polygonal domain \( D \) in \( \mathbb{R}^2 \) as \( t \downarrow 0 \):
\[
Z(D, t) \sim \frac{|D|}{4\pi t} - \frac{1}{4} \frac{|\partial D|}{(4\pi t)^{1/2}} + C,
\]
where the constant term, \( C \), was a complicated integral expression for each interior angle of \( D \). The constant term in the above expansion was found in the following simplified form by D.B. Ray and is stated in [14] as:
\[
\frac{\pi^2 - \beta_j^2}{24\pi^2},
\]
where \( \beta_j \) is an interior angle of the of the polygon. The formula (7) has since been improved in [2], where the remainder term is shown to be exponentially small.

Unlike the planar case, (where the trace expansion terminates after the constant term) the trace function for a geodesic spherical polygon admits a complete asymptotic expansion of the form:
\[
Z(\Omega, t) = \frac{|\Omega|}{4\pi t} - \frac{1}{4} \frac{|\partial \Omega|}{(4\pi t)^{1/2}} + \sum_{j=0}^{N} \{b_j t^{1/2} + i_j + v_j\} t^j + O(t^{N+1}),
\]
for any \( N \in \mathbb{N} \) and coefficients \( b_j, i_j \) and \( v_j \) which we state explicitly in Corollary 3. It is interesting to note in Example 4 the manner in which the angle contribution propagates through the asymptotic expansion.

The techniques we use extend those suggested by Kac in [12]. Away from the boundary of \( \Omega \) we approximate the heat kernel \( K_\Omega(\theta, \theta, t) \) by the heat kernel associated with \( e^{-\Delta_\Omega t} \). For points close to the boundary but away from a corner we
approximate $K_{\Omega}(\theta, \theta, t)$ by the heat kernel associated with $e^{-\Delta_{\Omega} t}$, where $\mathbb{H}^2$ is a hemi–sphere of $S^2$. Finally, for points close to a corner of angle $\gamma$ we approximate $K_{\Omega}(\theta, \theta, t)$ by the heat kernel associated with $e^{-\Delta_{\gamma} t}$, where $\gamma = (\gamma, \gamma, \pi)$ is the spherical triangle

$$(\gamma, \gamma, \pi) = \{ (\theta, \phi) \in S^2 : 0 < \phi < \gamma, 0 < \theta < \pi \}.$$  

(10)

often referred to as a lune of angle $\gamma$. For notational convenience we will denote the trace function for the lune $l_{\gamma} j$ by $Z(\gamma j, t)$, constructed from the heat kernel $K_j(\theta, \theta, t)$ associated with the heat semi–group $e^{-\Delta_{\Omega} t}$.

As one would expect, the approximations which we have described above give rise to interior, boundary and angle contributions in the expansion of $Z(\Omega, t)$. Consequently, we define

$$Z_{\text{int}}(\Omega, t) := \frac{|\Omega|}{4\pi} \text{tr}(e^{-\Delta_{\Omega} t}),$$

(11)

and

$$Z_{\text{bnd}}(\Omega, t) := \frac{|\partial \Omega|}{2\pi} \left\{ \text{tr}(e^{-\Delta_{\Omega} t}) - \frac{1}{2} \text{tr}(e^{-\Delta_{\gamma} t}) \right\}.$$  

(12)

For the angle contributions we set,

$$Z_{\text{ver}}(\Omega, t) := \frac{1}{2} \sum_{j=1}^{M} \{ Z(\gamma_j, t) - Z_{\text{int}}(\gamma_j, t) - Z_{\text{bnd}}(\gamma_j, t) \}.$$  

(13)

In Section 2 we prove the following important theorem, which formalises Kac’s principle of not feeling the boundary, as discussed above.

**Theorem 1.** Let $\Omega \subset S^2$ be an open connected set with a piecewise geodesic boundary $\partial \Omega$. There exist constants $\delta = \delta(\Omega) > 0$, $C > 0$ and $t_0 > 0$ such that for all $0 < t < t_0$,

$$|Z(\Omega, t) - Z_{\text{int}}(\Omega, t) - Z_{\text{bnd}}(\Omega, t) - Z_{\text{ver}}(\Omega, t)| \leq C t^2 e^{3/8 t}.$$  

The shifted Laplace–Beltrami operator acting on $L^2(\Omega)$ is defined as

$$\Delta_{\Omega}^{(s)} := \Delta_\Omega + \frac{1}{4} \text{Id},$$

(14)

where $\text{Id}$ is the identity operator. The coefficients in the asymptotic expansion of the shifted trace function

$$Z^{(s)}(\Omega, t) := \text{tr}(e^{-\Delta_{\Omega}^{(s)} t}),$$

(15)

are easier to compute than the coefficients in the expansion of $Z(\Omega, t)$. In Corollary 3 we obtain formulae for the coefficients in expansion (9) by using the coefficients in the asymptotic series of the shifted trace function and the expression

$$\text{tr}(e^{-\Delta_{\Omega} t}) = e^{t/4} \text{tr}(e^{-\Delta_{\Omega}^{(s)} t}).$$

(16)

To derive the formulae, in Theorem 2, for all the coefficients in the expansion as $t \downarrow 0$ of $Z^{(s)}(\Omega, t)$ we use the standard Bernoulli polynomials and the Bernoulli polynomials of higher order which we denote by $B_n(x)$ and $B_{n}^{(m)}(x \mid a_1 \cdots a_m)$ respectively. See [8] vol I for further details.
The Bernoulli numbers, $B_n$, are defined, in the standard manner, by the following generating function:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n, \quad |t| < 2\pi. \quad (17)$$

Also the Bernoulli polynomials are given by the following expansion:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n, \quad |t| < 2\pi, \quad (18)$$

from which the identities

$$B_n(x) = \sum_{j=0}^{n} \binom{n}{j} B_j x^{n-j}, \quad B_n(0) = B_n, \quad \frac{d}{dx} B_n(x) = nB_{n-1}(x), \quad (19)$$

are derived.

In addition, the Bernoulli polynomials of higher order are defined by the generating function,

$$\prod_{n=1}^{m} (e^{a_n t} - 1) = \sum_{n=0}^{\infty} B_n^{(m)}(x | a_1 \cdots a_m) \frac{t^n}{n!}, \quad (20)$$

provided $|t| < 2\pi/a_l$ where $a_l = \max\{|a_1|, \ldots, |a_m|\}$.

Finally, we let the function $c_k(\gamma)$ be given by

$$c_k : (0, 2\pi] \to \mathbb{R}, \quad (21)$$

where

$$c_k(\gamma) = \frac{1}{2} \sum_{j=1}^{k+1} a_j^{(k)} \frac{\pi^{2j} - \gamma^{2j}}{\pi \gamma^{2j-1}}, \quad (22)$$

with

$$a_j^{(k)} = \frac{(-1)^k}{2(k+1)!} \frac{1}{2k+1} \left(\frac{2k+2}{2j}\right) B_{2j} B_{2k+2-2j} \left(\frac{1}{2}\right). \quad (23)$$

In Section 3 we prove the main results which are Theorem 2 and Corollary 3 below:

**Theorem 2.** Let $\Omega$ be an open connected set in $S^2$ with a piecewise geodesic boundary $\partial \Omega$. Then as $t \downarrow 0$ and for any $N \in \mathbb{N}$,

$$Z^{(s)}(\Omega, t) = \frac{|\Omega|}{4\pi t} - \frac{1}{4} \frac{|\partial \Omega|}{(4\pi t)^{1/2}} + \sum_{k=0}^{N} \left( i_k^{(s)} + v_k^{(s)} \right) t^k + O(t^{N+1}), \quad (24)$$

where

$$i_k^{(s)} = (-1)^{(k+1)} B_{2k+2} \left(\frac{1}{2}\right) \frac{|\Omega|}{4\pi} \quad \text{and} \quad v_k^{(s)} = \sum_{j=1}^{M} c_j(\gamma_j). \quad (25)$$

**Corollary 3.** Let $\Omega$ be an open connected set in $S^2$ with a piecewise geodesic boundary $\partial \Omega$. Then as $t \downarrow 0$ and for any $N \in \mathbb{N}$,

$$Z(\Omega, t) = \frac{|\Omega|}{4\pi t} - \frac{1}{4} \frac{|\partial \Omega|}{(4\pi t)^{1/2}} + \sum_{j=0}^{N} \left( b_j t^{1/2} + i_j + v_j \right) t^j + O(t^{N+1}). \quad (26)$$
The boundary contributions are determined by:

\[ b_j = -\frac{|\partial \Omega|}{2\sqrt{\pi}4^{j} + (j + 1)!}. \]  
(27)

The interior contributions are determined by:

\[ i_j = \frac{1}{(j + 1)!} \sum_{k=0}^{j+1} (-1)^k B_{2k} \left( \frac{1}{4} \right) \left( 1 + \frac{1}{4} \right)^{j+1-k}. \]  
(28)

The angle contributions are determined by:

\[ v_j = \sum_{k=0}^{j} v_k(s) \left( \frac{1}{4} \right)^{j-k} \frac{1}{(j-k)!}. \]  
(29)

Example 4. For \( N = 5 \):

\[
Z(\Omega, t) = \frac{\Omega}{4\pi t} - \frac{1}{4} \left( \frac{\partial \Omega}{4\pi t} \right)^{1/2} \left\{ \pi^2 - \frac{\gamma_j}{12\pi \gamma_j} \right\} + \frac{\Omega}{12\pi} - \frac{|\partial \Omega| t^{1/2}}{32\sqrt{\pi}}
\]

\[ + \left( \frac{\pi^2 - \gamma_j}{36\pi \gamma_j} + \frac{\pi^4 - \gamma_j^2}{306\pi \gamma_j^3} \right) t + \frac{\Omega |t|}{60\pi} - \frac{|\partial \Omega| t^{3/2}}{256\sqrt{\pi}}
\]

\[ + \left( \frac{\pi^2 - \gamma_j}{180\pi \gamma_j} + \frac{\pi^4 - \gamma_j^2}{720\pi \gamma_j^3} + \frac{\pi^6 - \gamma_j^4}{2520\pi \gamma_j^5} \right) t^2 + \frac{\Omega |t|^2}{315\pi} - \frac{|\partial \Omega| t^{5/2}}{3072\sqrt{\pi}}
\]

\[ + \left( \frac{\pi^2 - \gamma_j}{945\pi \gamma_j} + \frac{\pi^4 - \gamma_j^2}{2160\pi \gamma_j^3} + \frac{\pi^6 - \gamma_j^4}{3780\pi \gamma_j^5} + \frac{\pi^8 - \gamma_j^6}{10080\pi \gamma_j^7} \right) t^3 + \frac{|\partial \Omega| t^{7/2}}{1260\pi} - \frac{|\partial \Omega| t^{5/2}}{49152\sqrt{\pi}}
\]

\[ + \left( \frac{\pi^2 - \gamma_j}{3780\pi \gamma_j} + \frac{\pi^4 - \gamma_j^2}{6480\pi \gamma_j^3} + \frac{\pi^6 - \gamma_j^4}{9072\pi \gamma_j^5} + \frac{\pi^8 - \gamma_j^6}{12096\pi \gamma_j^7} \right) t^4
\]

\[ + \frac{|\partial \Omega| t^{9/2}}{3465\pi} - \frac{|\partial \Omega| t^{7/2}}{983040\sqrt{\pi}}
\]

where the prime \( ' \) denotes half the sum over the angles; \( \{ \cdot \}' = \frac{1}{2} \sum_{j=1}^{M} \{ \cdot \}.

1. Heat Kernel Approximations

In order to make the approximations we have described in the introduction we use the following probabilistic representation for the heat kernel:

\[ K_{\Omega}(\theta, \theta, t) = K_{\mathbb{S}^2}(t) \cdot \text{Prob} \{ B(\tau) \in \Omega, 0 \leq \tau \leq t \mid B(0) = B(t) = \theta \}, \]

(30)
where $K_{S^2}(t)$ is the diagonal element of the heat kernel associated with $e^{-\Delta_{S^2}t}$. We note that $K_{S^2}(\theta, \theta, t) \equiv K_{S^2}(t)$ is independent of $\theta = (\theta, \phi)$. Indeed, it is easy to see that

$$K_{S^2}(t) \equiv \frac{1}{4\pi} Z(S^2, t).$$  \hfill (31)

The function $(B(t), t > 0; \mathbb{P}_0, \theta \in \Omega)$ is a Brownian motion associated to the parabolic operator $\partial_t - \Delta_{\Omega}$. The representation (31) has been used, for example, by Hsu [11] and Molchanov [15].

To approximate the heat kernel $K_{\Omega}(\theta, \theta, t)$ when $\theta$ is near a spherical corner of $\Omega$ by the heat kernel associated with $e^{-\Delta_{S^2}t}$ we have to isolate a region around the corner of the lune. Let $l_\beta$ be the lune of angle $\beta$.

If $0 < \beta < \pi$ then let $q \in l_\beta$ be a distance $\delta$ along the geodesic bisector of $\beta$ from one of the angles of the spherical triangle, for some $0 < \delta \ll \pi/2$. Form the spherical quadrilaterals $V_\beta(\delta)$, by extending from $q$ geodesics which are orthogonal to the sides of $l_\beta$. Denote the angles of $\Omega$ by $\gamma_1, \ldots, \gamma_M$ and around each angle form the region $V_{\gamma_j}(\delta)$ which we will denote by $V_{\gamma_j}(\delta)$.

If $\pi \leq \beta < 2\pi$ then bisect the angle $\beta$, so that $\beta/2 = \beta_1 = \beta_2$, and construct the quadrilaterals $V_{\beta_j}(\delta)$, $j = 1, 2$ around the bisected angle as described above. Then set $V_{\beta}(\delta) = V_{\beta_1}(\delta) \cup V_{\beta_2}(\delta)$.

We now use a similar construction to that in [2]:

$$\kappa = \frac{1}{2} \sup \left\{ \delta \mid V_j(\delta) \cap V_k(\delta) = \emptyset, j \neq k, \bigcup_{j=1}^{M} V_j(\delta) \subset \Omega \right\},$$ \hfill (32)

$$C_{\Omega}(\delta, \xi) = \left\{ \theta \in \Omega \mid d(\theta, \partial \Omega) < \delta, \theta \notin \bigcup_{j=1}^{M} V_j(\xi) \right\},$$ \hfill (33)

$$D_{\Omega}(\delta, \xi) = \left\{ \theta \in \Omega \mid \theta \notin C(\delta, \xi), \theta \notin \bigcup_{j=1}^{M} V_j(\xi) \right\}.$$ \hfill (34)

The proof of the following lemma follows from [11].

**Lemma 5.** For $\theta \in D_{\Omega}(\delta/2, \kappa)$ there exist constants $C > 0$ and $t_0 > 0$ such that for $0 < t < t_0$,

$$|K_{\Omega}(\theta, \theta, t) - K_{S^2}(\theta, \theta, t)| \leq \frac{C}{t} e^{-\delta^2/8t}.$$

Throughout the rest of this paper the parameter $t_0$ is that indicated in Lemma 5.

**Lemma 6.** For $\theta \in V_j(\kappa)$ there exists a constant $C > 0$ such that for $0 < t < t_0$,

$$|K_{\Omega}(\theta, \theta, t) - K_j(\theta, \theta, t)| \leq \frac{C}{t} e^{-\kappa^2/2t}.$$ \hfill (35)

**Lemma 7.** For $\theta \in C_{\Omega}(\delta/2, \kappa)$ there exists a constant $C > 0$ such that for $0 < t < t_0$,

$$|K_{\Omega}(\theta, \theta, t) - K_{S^2}(\theta, \theta, t)| \leq \frac{C}{t} e^{-\delta^2/8t}.$$ \hfill (36)
We define a “boundary layer” for the hemi–sphere as:

\[ B_{\alpha, \beta} := \{(r, \theta, \phi) \mid r = 1, \pi/2 - \alpha \leq \theta \leq \pi/2, 0 \leq \phi \leq \beta \}, \]

for \( 0 < \alpha \leq \pi/2 \) and \( 0 < \beta \leq 2\pi \).

**Lemma 8.** There exists a constant \( C > 0 \) such that for \( 0 < t < t_0 \)

\[
\left| \int_{B_{\alpha, 2\pi}} K_{\mathbb{H}^2}(\theta, \theta, t)d\theta - Z_{\text{int}}(B_{\alpha, 2\pi}, t) - Z_{\text{bnd}}(\mathbb{H}^2, t) \right| < \frac{C}{t} e^{-\alpha^2/4t}.
\]

**Proof.** We have

\[
\int_{B_{\alpha, 2\pi}} K_{\mathbb{H}^2}(\theta, \theta, t)d\theta = \int_{\mathbb{H}^2} K_{\mathbb{H}^2}(\theta, \theta, t)d\theta - \int_{\mathbb{H}^2/B_{\alpha, 2\pi}} K_{\mathbb{H}^2}(\theta, \theta, t)d\theta.
\]

However, for \( \theta \in \mathbb{H}^2 \setminus B_{\alpha, 2\pi} \) we have from Lemma 5 that

\[
|K_{\mathbb{H}^2}(\theta, \theta, t) - K_{\mathbb{H}^2}(\theta, \theta, t)| \leq \frac{C}{t} e^{-\alpha^2/4t}.
\]

Furthermore, since

\[
K_{\mathbb{H}^2}(\theta, \theta, t) = \frac{1}{4\pi} Z(S^2, t),
\]

we have

\[
\int_{B_{\alpha, 2\pi}} K_{\mathbb{H}^2}(\theta, \theta, t)d\theta = \text{tr}(e^{-\Delta_{\mathbb{H}^2}t}) - \int_{\mathbb{H}^2/B_{\alpha, 2\pi}} K_{\mathbb{H}^2}(\theta, \theta, t)d\theta + O(t^{-1} e^{-\alpha^2/4t}),
\]

and the result follows. \( \square \)

**Lemma 9.** For \( 0 < t < t_0 \),

\[
\int_{B_{\alpha, \beta}} K_{\mathbb{H}^2}(\theta, \theta, t)d\theta = \frac{\beta}{2\pi} \int_{B_{\alpha, 2\pi}} K_{\mathbb{H}^2}(\theta, \theta, t)d\theta.
\]

**Proof.** The result follows upon noting that the heat kernel \( K_{\mathbb{H}^2}(\theta, \theta, t) \) is independent of \( \theta \), where we recall that \( \theta = (\theta, \phi) \). \( \square \)

**Lemma 10.** There exist constants \( C > 0 \) and \( \delta > 0 \) such that for \( 0 < t < t_0 \),

\[
2 \int_{V_{\beta}(\kappa)} K_{l_\beta}(\theta, \theta, t)d\theta - \{Z(l_\beta, t) - Z_{\text{int}}(G, t) - Z_{\text{bnd}}(G, t)\} < \frac{C}{t} e^{-\delta^2/8t},
\]

where \( G = D_{l_\beta}(\delta/2, \kappa) \cup C_{l_\beta}(\delta/2, \kappa) \).

**Proof.** Let the lune \( l_\beta \) have angles \( \beta_1 = \beta_2 = \beta \). Choose \( \delta = \beta/2 \). Then

\[
l_\beta = G \cup V_1(\kappa) \cup V_2(\kappa).
\]

So by symmetry,

\[
2 \int_{V_{\beta}(\kappa)} K_{l_\beta}(\theta, \theta, t)d\theta = \int_{l_\beta} K_{l_\beta}(\theta, \theta, t)d\theta - \int_{G} K_{l_\beta}(\theta, \theta, t)d\theta.
\]

The result follows using Lemmas 5, 8 and 9. \( \square \)
Proof of Theorem 1. Put $\gamma = \min \{\gamma_j\}$. Choose $\delta = \delta(\Omega) > 0$ such that $\sin \delta = \sin(\gamma/2) \sin \kappa$, then

$$
\Omega = D_\Omega(\delta/2, \kappa) \bigcup_{j=1}^M C_\Omega(\delta/2, \kappa) \cup V_j(\kappa). \quad (44)
$$

From Lemmas 5, 6 and 7 we immediately obtain,

$$
\left| Z(\Omega, t) - \int_{D_\Omega(\delta/2, \kappa)} K_{S^2}(\theta, \theta, t) d\theta - \int_{C_\Omega(\delta/2, \kappa)} K_{H^2}(\theta, \theta, t) d\theta - \sum_{j=1}^M \int_{V_j(\kappa)} K_j(\theta, \theta, t) d\theta \right| \leq \frac{C}{t} e^{-\delta^2/8t}. \quad (45)
$$

Lemmas 8, 9 and 10 now give the desired result.

2. The Angle Contribution

The coefficients $b_j$ and $i_j$ given in Corollary 3 are relatively easy to compute using Theorem 2. The difficulty is presented by the angle coefficient $v_j$. To determine the angle contribution $v_j$ we first calculate the angle contribution for the shifted trace function $Z(s)(l_\beta, t)$ for a lune of angle $0 < \beta < 2\pi$, and then use the formula (16).

Lemma 11. For $m \in \mathbb{N}$, $n \in \mathbb{N}$ and $x \in \mathbb{R}$,

(i) $B_{n+1}^{(m)}(x|1) = (1 - n/m)B_n^{(m)}(x|1) + n(x/n - 1)B_{n-1}(x|1),$

(ii) $B_n^{(2)}(x|a) = \sum_{j=0}^n \binom{n}{j} a^j B_j B_{n-j}(x),$

(iii) $B_{2n}^{(2)}(x|a) = \sum_{j=0}^{2n} \binom{2n}{2j} (a^{2j-1}B_{2j} B_{2n-2j}(x) - (2n-1)B_{2n}(x) + (x-2n)B_{2n-1}(x).$

Proof. Part (I) Use the identities

$$
\frac{t^n e^{xt}}{(e^t - 1)^n} = \frac{t^n e^{xt}}{(e^t - 1)^{n-1}} + \frac{t^n e^{xt}}{(e^t - 1)^n}, \quad (45)
$$

and

$$
\frac{d}{dt} \left\{ \frac{t^n e^{xt}}{(e^t - 1)^n} \right\} = x \frac{t^n e^{xt}}{(e^t - 1)^{n-1}} + \frac{n}{t} \frac{t^n e^{xt}}{(e^t - 1)^n} - \frac{n}{t} \frac{t^n e^{xt}}{(e^t - 1)^{n+1}}, \quad (46)
$$

together with the definition of the Bernoulli polynomials.

Part (II) The polynomial $B_n^{(2)}(x|a)/n!$ is given by the coefficient of $t^n$ in the expansion as $t \downarrow 0$ of

$$
f(x, a, t) := \frac{at^2 e^{xt}}{(e^t - 1)(e^{at} - 1)}. \quad (47)$$
We write $f(x, a, t) = g(x, t)h(a, t)$ where
\[ g(x, t) = \frac{te^{xt}}{e^{at} - 1}, \quad \text{and} \quad h(a, t) = \frac{at}{e^{at} - 1}. \] (48)
But from (17) and (18) we see that for $|t| < \min\{2\pi, 2\pi/a\}$,
\[ g(x, t) = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad \text{and} \quad h(a, t) = \sum_{n=0}^{\infty} B_n \frac{(at)^n}{n!}. \] (49)
And the result follows by taking the product of the two sums in (49) and evaluating the coefficient of $t^n$.

**Part (III)** This follows by combining the results in Part (I) and Part (II). □

In the subsequent lemmas we will make use of the Euler–Maclaurin summation formula. We see from [18] that for a function $g \in C^M(b, c)$ with $b, c \in \mathbb{Z}$
\[ \sum_{j=b}^{c} g(j) = \int_{b}^{c} g(x)dx + \sum_{k=1}^{M} \frac{B_k}{k!} [g^{(k-1)}(c) - g^{(k-1)}(b)] + S_M(g), \] (50)
where
\[ S_M(g) = -\sum_{j=b}^{c} \int_{j}^{j+1} \frac{B_M(j + 1 - y)}{M!} g^M(y)dy. \] (51)
The complementary error function $\text{erfc}(-)$ is defined by, (see for example [13]),
\[ \text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_{z}^{\infty} e^{-x^2} dx. \] (52)

**Lemma 12.** Let $g_n(x, a, t) = e^{-(ax+n+1/2)^2t}$ with $a > 0$ and $t > 0$. Then as $t \downarrow 0$
and for any $N \in \mathbb{N}$,
\[ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} g_n(m, a, t) \]
\[ = \sum_{n=0}^{\infty} \frac{\sqrt{\pi}}{2a\sqrt{t}} \text{erfc}\left(\left(n + \frac{1}{2}\right)\sqrt{t}\right) + \frac{1}{2} \sum_{n=0}^{\infty} e^{-(n+1/2)^2t} + \sum_{k=0}^{N} a_k t^k + O(t^{N+1}), \] (53)
for some coefficients $a_k \in \mathbb{R}$.

**Proof.** Since, as a function of $x$, $g_n(x, a, t) \in C^\infty(0, \infty)$ we can apply the Euler–Maclaurin summation formula:
\[ \sum_{n=0}^{\infty} g_n(m, a, t) \]
\[ = \frac{\sqrt{\pi}}{2a\sqrt{t}} \text{erfc}\left(\left(n + 1/2\right)t^{1/2}\right) + \frac{1}{2} e^{-(n+1/2)^2t} - \sum_{k=2}^{M} \frac{B_k}{k!} g_n^{(k-1)}(0, a, t) + S_M(g_n), \] (54)
for some $M \in \mathbb{N}$. Since $B_{2j+1} = 0$ for $j = 1, 2, \ldots$ we will assume that $M$ is even. Thus, we need to determine the odd derivatives, with respect to $x$, of the function $g_n(x,a,t)$. An application of Leibniz’s formula yields,

\[
g_n^{(2k+1)}(0,a,t) = a^{2k+1} \sum_{j=0}^{2k} \binom{2k+1}{2j} (2j-1)!! (n + \frac{j}{2})^{2k+1-2j} e^{-(n+1/2)^2 t}.
\]  \tag{55}

From Mulholland [16] we see that as $t \downarrow 0$,

\[
2^{k+1} \sum_{n=0}^{\infty} (n+1/2)^{2k+1} e^{-(k+1/2)^2 t} = k! + \sum_{l=0}^{R} a_l t^l + O(t^{R+1}),
\]  \tag{56}

for any $R \in \mathbb{N}$ and some known coefficients $a_l \in \mathbb{R}$. Thus, combining (55) and (56) gives as $t \downarrow 0$

\[
\sum_{n=0}^{\infty} g_n^{(2k+1)}(0,a,t) = t^k \sum_{j=0}^{R} c_j t^j + O(t^{k+R+1}),
\]  \tag{57}

for some coefficients $c_j \in \mathbb{R}$. For the remainder term,

\[
S_M(g_n) = -\sum_{j=0}^{\infty} \int_{0}^{1} B_M(z) M! g^{(M)}(j+1-z,a,t) dz,
\]  \tag{58}

we note that over the interval $[0,1]$ the uniform bound

\[
\left| \frac{B_M(z)}{M!} \right| < C
\]  \tag{59}

holds for some constant $C > 0$. Thus,

\[
|S_M(g_n)| < C \int_{0}^{\infty} \left| g_n^{(M)}(z,a,t) \right| dz \leq C|g_n^{(M-1)}(0,a,t)|,
\]  \tag{60}

and so

\[
\sum_{n=0}^{\infty} |S_M(g_n)| < C \sum_{n=0}^{\infty} |g_n^{(M-1)}(0,a,t)|.
\]  \tag{61}

If we choose $M = 2N + 4$, then

\[
\left| \sum_{n=0}^{\infty} S_M(g_n) \right| \leq \sum_{n=0}^{\infty} |S_M(g_n)| < C \sum_{n=0}^{\infty} |g_n^{(M-1)}(0,a,t)| < Ct^{N+1},
\]  \tag{62}

where we have used the expansion (57) to obtain the third inequality above. □
Lemma 13. Let \( f(x, a, t) = \sqrt{\frac{\pi}{2a\sqrt{t}}} \text{erfc}( (x + \frac{1}{2})\sqrt{t}) \) with \( a > 0 \) and \( t > 0 \). Then as \( t \downarrow 0 \) and for any \( N \in \mathbb{N} \),

\[
\sum_{n=0}^{\infty} f(n, a, t) = \frac{1}{2at} + \sum_{k=0}^{N} c_k t^k + O(t^{N+1}),
\]

for some coefficients \( c_k \in \mathbb{R} \).

Proof. Since, as a function of \( x \), \( f(x, a, t) \in C^\infty(0, \infty) \), we can again apply the Euler–Maclaurin summation formula:

\[
\sum_{n=0}^{\infty} f(n,a,t) = \frac{\sqrt{\pi}}{2a\sqrt{t}} \int_0^\infty \text{erfc} \left( (x + \frac{1}{2})\sqrt{t} \right) dx + \frac{1}{2} \text{erfc}(\sqrt{t}/2) + \frac{\sqrt{\pi}}{2a\sqrt{t}} \sum_{k=0}^{M/2} \frac{B_k}{k!} f^{(k-1)}(0,a,t) + S_M(f)
\]

for some \( M, M' \in \mathbb{N} \) and where \( h(x, t) = e^{-(x^2 + x^2)t} \). Using exactly the same approach as in Lemma 12 gives the desired result. \( \square \)

Proposition 14. Let \( l_\beta \) be a lune of angle \( \beta \). Then as \( t \downarrow 0 \) and for any \( N \in \mathbb{N} \),

\[
Z(s)(l_\beta, t) = \frac{\beta^2}{2\pi t} - \frac{\sqrt{\pi}}{4\sqrt{t}} \sum_{j=0}^{N} c_j^{(s)} t^j + O(t^{N+1}),
\]

for some coefficients \( c_j^{(s)} \in \mathbb{R} \).

Proof. In [10], Gromes gives the Dirichlet eigenvalues for a lune of angle \( 0 < \beta \leq 2\pi \) as

\[
\lambda_{n,m}(\lambda_{n,m} + 1) \quad \text{where} \quad \lambda_{n,m} = \frac{\pi m}{\beta} + n,
\]

with \( m = 1, 2, \cdots \) and \( n = 0, 1, \cdots \). Hence,

\[
Z^{(s)}(l_\beta, t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} e^{-(m\pi/\beta + n + 1/2)^2 t},
\]

\[
= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} e^{-(m\pi/\beta + n + 1/2)^2 t} - \sum_{n=0}^{\infty} e^{-(n+1/2)^2 t},
\]

where we have put \( a = \pi/\beta \). First we note from [20] the formula for the theta function:

\[
\sum_{n=-\infty}^{\infty} e^{-n^2 t^2} = \frac{\sqrt{\pi}}{t} \sum_{n=-\infty}^{\infty} e^{-n^2 \pi^2 / t^2},
\]
valid for $t > 0$, from which it easily follows by considering even and odd summations that
\[
\left| \sum_{n=0}^{\infty} e^{-(n+1/2)^2 t} - \frac{\sqrt{\pi}}{2\sqrt{t}} \right| < C e^{-1/t},
\] for some constant $C > 0$. Combining Lemmas 12 and 13 with expression (70) gives the result.

Having established the existence in Proposition 14 of the asymptotic expansion as $t \downarrow 0$ of $Z^{(s)}(l_\beta, t)$, we now determine explicitly the coefficients $c_j^{(s)}$.

**Lemma 15.** Let $l_\beta$ be a lune of angle $\beta$. Then as $t \downarrow 0$ and for any $N \in \mathbb{N}$ there exists a constant $C > 0$ such that,
\[
\left| Z^{(s)}(l_\beta, t) - \frac{\beta^2}{2\pi t} + \frac{\sqrt{\pi}}{4\sqrt{t}} - \sum_{j=0}^{N} c_j^{(s)} t^j \right| < C t^{N+1},
\] where
\[
c_k^{(s)} = \frac{(-1)^k}{2a(k+1)!} \frac{1}{2k+1} \sum_{j=0}^{k+1} \left( \frac{2k+2}{2j} \right) (a^2)^j - 1 B_{2j} B_{2k+2-2j} \left( \frac{1}{2} \right) + \frac{(-1)^{k+1} B_{2k+2} \left( \frac{1}{2} \right)}{2a} \left( \frac{1}{k+1} \right)!.
\] and $a = \pi/\beta$.

**Proof.** From Proposition 14 we note that
\[
Z^{(s)}(l_\beta, t) = \frac{\beta^2}{2\pi t} + \frac{\sqrt{\pi}}{4\sqrt{t}} - \sum_{j=0}^{N} c_j^{(s)} t^j + O(t^{N+1}),
\] for some coefficients $c_j^{(s)}$. We compute the coefficients $c_j^{(s)}$ by determining the residues of the function $\Gamma(\eta) \zeta_{\delta}(\beta, \eta)$ at $\eta = -k$ for $k = 0, 1, \cdots$. The zeta function $\zeta_{\delta}(\beta, \eta)$ is given for $\Re(\eta) > 1$ by,
\[
\zeta_{\delta}(\beta, \eta) = \frac{1}{\Gamma(\eta)} \int_0^\infty t^{\eta-1} Z^{(s)}(l_\beta, t) \, dt,
\] and
\[
= \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(ma + n + 1/2)^{2\eta}}.
\] However, we recognise that
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(ma + n + 1/2)^{2\eta}} = \zeta_2(2\eta, 1/2|a),
\] where $\zeta_2(2\eta, 1/2|a)$ is the Barnes’ zeta function introduced in [1]. But from Proposition 14, in particular the expansion (68), we note that extending the $m$ summation as we have in (76) does not change the residues of the zeta function at $\eta = -k$, that is:
\[
\text{Res}_{\eta=-k} \zeta_{\delta}(\beta, \eta) = \text{Res}_{\eta=-k} \zeta_2(2\eta, 1/2|a).
\]
In terms of the generalised Bernoulli polynomials of higher order we see from \([1]\) that
\[\zeta_s(\beta, -k) = \frac{B_{2k+2}(1/2|a)}{a(2k+2)(2k+1)} + O\left(t^{N+1}\right).\]  
(78)
The result now follows using Lemma 11 Part(III). □

3. Proof of the Main Theorem

The proof of Lemma 16 below is an application of the theorem of Mulholland \([16]\).

Lemma 16. Let \(\Omega\) be a geodesic spherical polygon in \(S^2\). Then as \(t \downarrow 0\) and for any \(N \in \mathbb{N}\),
\[Z^{(s)}(\Omega, t) = \left|\Omega\right| \frac{1}{4\pi} \left\{ 1 + \sum_{k=0}^{N} i_k^{(s)} t^k + O(t^{N+1}) \right\},\]  
(79)
where
\[i_k^{(s)} = \frac{(-1)^k B_{2k+2} \left(\frac{1}{2}\right)}{(k+1)!}.\]  
(80)

Lemma 17. Let \(\Omega\) be a geodesic spherical polygon in \(S^2\). Then as \(t \downarrow 0\),
\[Z^{(s)}_{\text{bnd}}(\Omega, t) = -\frac{1}{4} \left|\partial \Omega\right| + O(e^{-1/t}).\]  
(81)

Proof. From formula (12) and the corresponding eigenvalues we have
\[Z^{(s)}_{\text{bnd}}(\Omega, t) = -\frac{1}{4\pi} \left|\partial \Omega\right| \sum_{k=0}^{\infty} e^{-(k+1/2)^2 t}.\]  
(82)
But using the formula (70) we get,
\[\sum_{k=0}^{\infty} e^{-(k+1/2)^2 t} = \frac{\sqrt{\pi}}{2\sqrt{t}} + O(e^{-1/t}).\]  
(83)
and the result follows. □

Lemma 18. Let \(l_\beta\) be a lune of angle \(\beta\). Then there exist constants \(C > 0\) and \(N \in \mathbb{N}\) such that as \(t \downarrow 0\),
\[\left| Z^{(s)}(l_\beta, t) - Z^{(s)}_{\text{int}}(l_\beta, t) - Z^{(s)}_{\text{bnd}}(l_\beta, t) - \sum_{j=0}^{N} a_j^{(s)} t^j \right| < C t^{N+1},\]  
(84)
where
\[a_k^{(s)} = \frac{(-1)^k}{2a(k+1)!} \frac{1}{2k+1} \sum_{j=0}^{k+1} \binom{2k+2}{2j} (a^{2j}-1) B_{2j} B_{2k+2-2j} \left(\frac{1}{2}\right),\]  
(85)
and \(a = \pi/\beta\).
Proof. From Lemma 15 we see that
\[ Z^{(s)}(t, t) = \frac{\beta}{2\pi t} - \frac{1}{4\sqrt{t}} + \sum_{j=0}^{N} c_j^{(s)} t^j + O(t^{N+1}), \]  
(86)
where the coefficients \( c_j^{(s)} = \frac{i^j}{j!} + a_j^{(s)} \). Combining the expansions from Lemmas 16 and 17 with (86) gives the desired result. \( \square \)

Proof of Theorem 2. Using Theorem 1 we see that we can write,
\[ |Z^{(s)}(\Omega, t) - Z_{\text{int}}^{(s)}(\Omega, t) - Z_{\text{bnd}}^{(s)}(\Omega, t) - Z_{\text{ver}}^{(s)}(\Omega, t)| < \frac{C}{t} e^{-\kappa/t}, \]  
(87)
for some \( C > 0 \) and \( \kappa > 0 \). The result follows upon an application of Lemmas 16, 17 and 18.

Proof of Corollary 3. The proof easily follows using the formulae
\[ Z(\Omega, t) = e^{t/4} Z^{(s)}(\Omega, t) \quad \text{and} \quad e^{x} = \sum_{j=0}^{\infty} \frac{x^j}{j!}, \]  
(88)
and the expansion established in Theorem 2.

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References

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