GENERALIZED POPOVICIU FUNCTIONAL EQUATIONS IN BANACH MODULES OVER A $C^*$–ALGEBRA AND APPROXIMATE ALGEBRA HOMOMORPHISMS

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1. Introduction

Let $E_1$ and $E_2$ be Banach spaces with norms $\| \cdot \|$ and $\| \cdot \|$, respectively. Consider $f : E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\varepsilon \geq 0$ and $p \in [0, 1)$ such that

$$
\| f(x + y) - f(x) - f(y) \| \leq \varepsilon (\|x\|^p + \|y\|^p)
$$

for all $x, y \in E_1$. Th.M. Rassias [11] showed that there exists a unique $\mathbb{R}$–linear mapping $T : E_1 \to E_2$ such that

$$
\| f(x) - T(x) \| \leq \frac{2\varepsilon}{2 - 2p} \|x\|^p
$$

for all $x \in E_1$.

Throughout this paper, let $A$ be a unital $C^*$–algebra with norm $| \cdot |$, $U(A)$ the unitary group of $A$, $A_{sa}$ the set of self-adjoint elements in $A$, $A_1 = \{ a \in A \mid |a| = 1 \}$, and $A_1^+$ the set of positive elements in $A_1$. Let $A\mathcal{B}$ and $A\mathcal{C}$ be left Banach $A$–modules with norms $\| \cdot \|$ and $\| \cdot \|$, respectively. Let $n$ and $k$ be integers with $2 \leq k \leq n - 1$.

Recently, T. Trif [19] generalized the Popoviciu functional equation

$$
3f \left( \frac{x + y + z}{3} \right) + f(x) + f(y) + f(z) = 2 \left( f \left( \frac{x + y}{2} \right) + f \left( \frac{y + z}{2} \right) + f \left( \frac{z + x}{2} \right) \right).
$$

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Lemma A. [19, Theorem 2.1] Let $V$ and $W$ be vector spaces. A mapping $f: V \to W$ with $f(0) = 0$ satisfies the functional equation

$$n \cdot n^{-2} C_{k-2} f \left( \frac{x_1 + \cdots + x_n}{n} \right) + n^{-2} C_{k-1} \sum_{i=1}^{n} f(x_i) = k \cdot \sum_{1 \leq i_1 < \cdots < i_k \leq n} f \left( \frac{x_{i_1} + \cdots + x_{i_k}}{k} \right)$$

(A)

for all $x_1, \ldots, x_n \in V$ if and only if the mapping $f: V \to W$ satisfies the additive Cauchy equation $f(x + y) = f(x) + f(y)$ for all $x, y \in V$.

The following is useful to prove the stability of the functional equation (A).

Lemma B. [10, Theorem 1] Let $a \in A$ and $|a| < 1 - \frac{2}{m}$ for some integer $m$ greater than 2. Then there are $m$ elements $u_1, \ldots, u_m \in U(A)$ such that $ma = u_1 + \cdots + u_m$.

The main purpose of this paper is to prove the Hyers–Ulam–Rassias stability of the functional equation (A) in Banach modules over a unital $C^*$–algebra, and to prove the Hyers–Ulam–Rassias stability of algebra homomorphisms between Banach algebras associated with the functional equation (A).

2. Stability of Generalized Popoviciu Functional Equations in Banach Modules over a $C^*$–Algebra Associated with its Unitary Group

We are going to prove the Hyers–Ulam–Rassias stability of the functional equation (A) in Banach modules over a unital $C^*$–algebra associated with its unitary group.

For a given mapping $f: A \mathcal{B} \to A \mathcal{C}$ and a given $a \in A$, we set

$$D_a f(x_1, \ldots, x_n) := n \cdot n^{-2} C_{k-2} a f \left( \frac{x_1 + \cdots + x_n}{n} \right) + n^{-2} C_{k-1} \sum_{i=1}^{n} a f(x_i) - k \cdot \sum_{1 \leq i_1 < \cdots < i_k \leq n} f \left( \frac{a x_{i_1} + \cdots + a x_{i_k}}{k} \right)$$

for all $x_1, \ldots, x_n \in A \mathcal{B}$.

Theorem 2.1. Let $q = \frac{k(n-1)}{n-k}$ and $r = -\frac{k}{n-k}$. Let $f: A \mathcal{B} \to A \mathcal{C}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi: A \mathcal{B}^n \to [0, \infty)$ such that

$$\varphi(x_1, \ldots, x_n) := \sum_{j=0}^{\infty} t^{-j} \varphi(q^j x_1, \ldots, q^j x_n)$$

$$< \infty \|D_u f(x_1, \ldots, x_n)\| \leq \varphi(x_1, \ldots, x_n) \quad (2.1)$$

for all $u \in U(A)$ and all $x_1, \ldots, x_n \in A \mathcal{B}$. Then there exists a unique $A$–linear mapping $T: A \mathcal{B} \to A \mathcal{C}$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{k \cdot n^{-1} C_{k-1}} \varphi(q x, r x, \cdots, r x) \quad (2.2)$$

for all $x \in A \mathcal{B}$. 
Proof. Put \( u = 1 \in \mathcal{U}(A) \). By [19, Theorem 3.1], there exists a unique additive mapping \( T : \mathcal{A} \mathcal{B} \to \mathcal{A} \mathcal{C} \) satisfying (2.i). The additive mapping \( T : \mathcal{A} \mathcal{B} \to \mathcal{A} \mathcal{C} \) was defined by

\[
T(x) = \lim_{j \to \infty} q^{-j}f(q^j x)
\]

for all \( x \in \mathcal{A} \mathcal{B} \).

By the assumption, for each \( u \in \mathcal{U}(A) \),

\[
q^{-j}\|D_u f(q^j x, \ldots, q^j x)\| \leq q^{-j} \varphi(q^j x, \ldots, q^j x)
\]

for all \( x \in \mathcal{A} \mathcal{B} \), and

\[
q^{-j}\|D_u f(q^j x, \ldots, q^j x)\| \to 0
\]

as \( n \to \infty \) for all \( x \in \mathcal{A} \mathcal{B} \). So

\[
D_u T(x, \ldots, x) = \lim_{j \to \infty} q^{-j}D_u f(q^j x, \ldots, q^j x) = 0
\]

for all \( u \in \mathcal{U}(A) \) and all \( x \in \mathcal{A} \mathcal{B} \). Hence

\[
D_u T(x, \ldots, x) = n \cdot n - 2 C_{k - 2} u T(x) + n \cdot n - 2 C_{k - 1} u T(x) - k \cdot n C_k T(ux) = 0
\]

for all \( u \in \mathcal{U}(A) \) and all \( x \in \mathcal{A} \mathcal{B} \). So

\[
u T(x) = T(ux)
\]

for all \( u \in \mathcal{U}(A) \) and all \( x \in \mathcal{A} \mathcal{B} \).

Now let \( a \in A \) (\( a \neq 0 \)) and \( M \) an integer greater than \( 4|a| \). Then

\[
\frac{|a|}{M} = \frac{1}{M}|a| < \frac{|a|}{4|a|} = \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}.
\]

By Lemma B, there exist three elements \( u_1, u_2, u_3 \in \mathcal{U}(A) \) such that \( \frac{3a}{M} = u_1 + u_2 + u_3 \). And \( T(x) = T \left( \frac{3}{2} x \right) = 3 T \left( \frac{1}{3} x \right) \) for all \( x \in \mathcal{A} \mathcal{B} \). So \( T \left( \frac{1}{3} x \right) = \frac{1}{3} T(x) \) for all \( x \in \mathcal{A} \mathcal{B} \). Thus

\[
T(ax) = T \left( \frac{M}{3} \cdot \frac{3a}{M} x \right) = M \cdot T \left( \frac{1}{3} \cdot \frac{3a}{M} x \right) = \frac{M}{3} T \left( \frac{3a}{M} x \right)
\]

\[
= \frac{M}{3} T(u_1 x + u_2 x + u_3 x) = \frac{M}{3} (T(u_1 x) + T(u_2 x) + T(u_3 x))
\]

\[
= \frac{M}{3} (u_1 + u_2 + u_3) T(x) = \frac{M}{3} \cdot 3 \cdot \frac{a}{M} T(x)
\]

\[
= a T(x)
\]

for all \( x \in \mathcal{A} \mathcal{B} \). Obviously, \( T(0x) = 0 T(x) \) for all \( x \in \mathcal{A} \mathcal{B} \). Hence

\[
T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)
\]

for all \( a, b \in A \) and all \( x, y \in \mathcal{A} \mathcal{B} \). So the unique additive mapping \( T : \mathcal{A} \mathcal{B} \to \mathcal{A} \mathcal{C} \) is an \( A \)-linear mapping, as desired. \( \square \)

Applying the unital \( C^* \)-algebra \( C \) to Theorem 2.1, one can obtain the following.
Corollary 2.2. Let $E_1$ and $E_2$ be complex Banach spaces with norms $\| \cdot \|$ and $\| \cdot \|$, respectively. Let $q = \frac{k(n-1)}{n-k}$ and $r = -\frac{k}{n-k}$. Let $f : E_1 \to E_2$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : E_1^n \to [0, \infty)$ such that

$$\varphi(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} q^{-j} \varphi(q^j x_1, \cdots, q^j x_n)$$

for all $\lambda \in \mathbb{T}^1 := \{ \lambda \in \mathbb{C} \mid |\lambda| = 1 \}$ and all $x_1, \cdots, x_n \in E_1$. Then there exists a unique $\mathbb{C}$–linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{k \cdot n^{-1} C_{k-1}} \varphi(qx, rx, \cdots, rx)$$

for all $x \in E_1$.

Theorem 2.3. Let $q = \frac{k(n-1)}{n-k}$ and $r = -\frac{k}{n-k}$. Let $f : A\mathcal{B} \to A\mathcal{C}$ be a continuous mapping with $f(0) = 0$ for which there exists a function $\varphi : A\mathcal{B}^n \to [0, \infty)$ satisfying (2.i) such that

$$\|D_u f(x_1, \cdots, x_n)\| \leq \varphi(x_1, \cdots, x_n)$$

for all $u \in U(A)$ and all $x_1, \cdots, x_n \in A\mathcal{B}$. If the sequence $\{q^{-j} f(q^j x)\}$ converges uniformly on $A\mathcal{B}$, then there exists a unique continuous $A$–linear mapping $T : A\mathcal{B} \to A\mathcal{C}$ satisfying (2.ii).

Proof. Put $u = 1 \in U(A)$. By Theorem 2.1, there exists a unique $A$–linear mapping $T : A\mathcal{B} \to A\mathcal{C}$ satisfying (2.ii). By the continuity of $f$, the uniform convergence and the definition of $T$, the $A$–linear mapping $T : A\mathcal{B} \to A\mathcal{C}$ is continuous, as desired. \hfill \Box

Theorem 2.4. Let $q = \frac{k(n-1)}{n-k}$ and $r = -\frac{1}{n-k}$. Let $f : A\mathcal{B} \to A\mathcal{C}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : A\mathcal{B}^n \to [0, \infty)$ such that

$$\varphi(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} q^{-j} \varphi(q^j x_1, \cdots, q^j x_n)$$

$$\|D_u f(x_1, \cdots, x_n)\| \leq \varphi(x_1, \cdots, x_n)$$

for all $u \in U(A)$ and all $x_1, \cdots, x_n \in A\mathcal{B}$. Then there exists a unique $A$–linear mapping $T : A\mathcal{B} \to A\mathcal{C}$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{n^{-2} C_{k-1}} \varphi(x, rx, \cdots, rx)$$

for all $x \in A\mathcal{B}$.

Proof. Put $u = 1 \in U(A)$. By [19, Theorem 3.2], there exists a unique additive mapping $T : A\mathcal{B} \to A\mathcal{C}$ satisfying (2.iv). The additive mapping $T : A\mathcal{B} \to A\mathcal{C}$ was defined by

$$T(x) = \lim_{j \to \infty} q^j f(q^{-j} x)$$

for all $x \in A\mathcal{B}$.

By the assumption, for each $u \in U(A)$,

$$q^j \|D_u f(q^{-j} x, \cdots, q^{-j} x)\| \leq q^j \varphi(q^{-j} x, \cdots, q^{-j} x)$$
for all \( x \in A \mathcal{B} \), and

\[
q^{1} \| D_{u} f(q^{-1} x, \ldots, q^{-j} x) \| \to 0
\]

as \( n \to \infty \) for all \( x \in A \mathcal{B} \). So

\[
D_{u} T(x, \ldots, x) = \lim_{j \to \infty} q^{j} D_{u} f(q^{-j} x, \ldots, q^{-j} x) = 0
\]

for all \( u \in U(A) \) and all \( x \in A \mathcal{B} \). Hence

\[
D_{u} T(x, \ldots, x) = n \cdot n_{2} C_{k-2} uT(x) + n \cdot n_{2} C_{k-1} uT(x) - k \cdot n C_{k} T(u x) = 0
\]

for all \( u \in U(A) \) and all \( x \in A \mathcal{B} \). So

\[
uT(x) = T(u x)
\]

for all \( u \in U(A) \) and all \( x \in A \mathcal{B} \).

The rest of the proof is the same as the proof of Theorem 2.1. \( \square \)


Given a locally compact abelian group \( G \) and a multiplier \( \omega \) on \( G \), one can associate to them the twisted group \( C^{*} \)–algebra \( C^{*}(G, \omega) \). \( C^{*}(\mathbb{Z}^{m}, \omega) \) is said to be a noncommutative torus of rank \( m \) and denoted by \( A_{\omega} \). The multiplier \( \omega \) determines a subgroup \( S_{\omega} \) of \( G \), called its symmetry group, and the multiplier is called totally skew if the symmetry group \( S_{\omega} \) is trivial. And \( A_{\omega} \) is called completely irrational if \( \omega \) is totally skew (see [2]). It was shown in [2] that if \( G \) is a locally compact abelian group and \( \omega \) is a totally skew multiplier on \( G \), then \( C^{*}(G, \omega) \) is a simple \( C^{*} \)–algebra.

It was shown in [3, Theorem 1.5] that if \( A_{\omega} \) is a simple noncommutative torus then \( A_{\omega} \) has real rank 0, where “real rank 0” means that the set of invertible self-adjoint elements is dense in the set of self-adjoint elements (see [3], [6]).

From now on, assume that \( \varphi : A \mathcal{B}^{n} \to \left[ 0, \infty \right) \) is a function satisfying (2.i), and that \( q = \frac{k(n-1)}{n} \) and \( r = -\frac{k}{n-1} \).

We are going to prove the Hyers–Ulam–Rassias stability of the functional equation (A) in Banach modules over a unital \( C^{*} \)–algebra.

**Theorem 3.1.** Let \( q \) be not an integer. Let \( f : A \mathcal{B} \to A \mathcal{C} \) be a mapping with \( f(0) = 0 \) such that

\[
\| D_{1} f(x_{1}, \ldots, x_{n}) \| \leq \varphi(x_{1}, \ldots, x_{n})
\]

for all \( x_{1}, \ldots, x_{n} \in A \mathcal{B} \). Then there exists a unique additive mapping \( T : A \mathcal{B} \to A \mathcal{C} \) satisfying (2.ii). Further, if \( f(\lambda x) \) is continuous in \( \lambda \in \mathbb{R} \) for each fixed \( x \in A \mathcal{B} \), then the additive mapping \( T : A \mathcal{B} \to A \mathcal{C} \) is \( \mathbb{R} \)–linear.

**Proof.** By the same reasoning as the proof of Theorem 2.1, there exists a unique additive mapping \( T : A \mathcal{B} \to A \mathcal{C} \) satisfying (2.ii).

Assume that \( f(\lambda x) \) is continuous in \( \lambda \in \mathbb{R} \) for each fixed \( x \in A \mathcal{B} \). By the assumption, \( q \) is a rational number which is not an integer. The additive mapping \( T \) given above is similar to the additive mapping \( T \) given in the proof of [11, Theorem]. By the same reasoning as the proof of [11, Theorem], the additive mapping \( T : A \mathcal{B} \to A \mathcal{C} \) is \( \mathbb{R} \)–linear. \( \square \)
Theorem 3.2. Let $q$ be not an integer. Let $f : A\mathcal{B} \to A\mathcal{C}$ be a continuous mapping with $f(0) = 0$ such that

$$\|D_a f(x_1, \cdots, x_n)\| \leq \varphi(x_1, \cdots, x_n)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x_1, \cdots, x_n \in A\mathcal{B}$. If the sequence $\{q^j f(q^j x)\}$ converges uniformly on $A\mathcal{B}$, then there exists a unique continuous $A$–linear mapping $T : A\mathcal{B} \to A\mathcal{C}$ satisfying (2.ii).

Proof. Put $a = 1 \in A_1^+$. By Theorem 3.1, there exists a unique $\mathbb{R}$–linear mapping $T : A\mathcal{B} \to A\mathcal{C}$ satisfying (2.ii). By the continuity of $f$ and the uniform convergence, the $\mathbb{R}$–linear mapping $T : A\mathcal{B} \to A\mathcal{C}$ is continuous.

By the same reasoning as the proof of Theorem 2.1,

$$T(ax) = aT(x)$$

for all $a \in A_1^+ \cup \{i\}$.

For any element $a \in A$, $a = \frac{a + a^*}{2} + i\frac{a - a^*}{2i}$, and $\frac{a + a^*}{2}$ and $\frac{a - a^*}{2i}$ are self–adjoint elements, furthermore, $a = \frac{a + a^*}{2} + i\frac{a - a^*}{2i} = i(\frac{a - a^*}{2i})^+$, where $(\frac{a + a^*}{2})^+ = (\frac{a - a^*}{2i})^+$ are positive elements (see [4, Lemma 38.8]). So

$$T(ax) = T\left(\left(\frac{a + a^*}{2}\right)^+ x - \left(\frac{a + a^*}{2}\right)^- x + i\left(\frac{a - a^*}{2i}\right)^+ x - i\left(\frac{a - a^*}{2i}\right)^- x\right)$$

$$= \left(\frac{a + a^*}{2}\right)^+ T(x) + \left(\frac{a + a^*}{2}\right)^- T(-x) + i\left(\frac{a - a^*}{2i}\right)^+ T(ix)$$

$$+ \left(\frac{a - a^*}{2i}\right)^- T(-ix)$$

$$= \left(\frac{a + a^*}{2}\right)^+ T(x) - \left(\frac{a + a^*}{2}\right)^- T(x) + i\left(\frac{a - a^*}{2i}\right)^+ T(x)$$

$$- i\left(\frac{a - a^*}{2i}\right)^- T(x)$$

$$= aT(x)$$

for all $a \in A$ and all $x \in A\mathcal{B}$. Hence

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in A\mathcal{B}$, as desired. 

Theorem 3.3. Let $A$ be a unital $C^*$–algebra of real rank 0, and $q$ not an integer. Let $f : A\mathcal{B} \to A\mathcal{C}$ be a continuous mapping with $f(0) = 0$ such that

$$\|D_a f(x_1, \cdots, x_n)\| \leq \varphi(x_1, \cdots, x_n)$$
for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x_1, \ldots, x_n \in \mathcal{A}\mathcal{B}$. If the sequence $\{q^{-j}f(q^jx)\}$ converges uniformly on $\mathcal{A}\mathcal{B}$, then there exists a unique continuous $\mathcal{A}$–linear mapping $T : \mathcal{A}\mathcal{B} \to \mathcal{A}\mathcal{C}$ satisfying $(2.ii)$.

**Proof.** By the same reasoning as the proof of Theorem 3.2, there exists a unique continuous $\mathbb{R}$–linear mapping $T : \mathcal{A}\mathcal{B} \to \mathcal{A}\mathcal{C}$ satisfying $(2.ii)$, and

$$T(ax) = aT(x)$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x \in \mathcal{A}\mathcal{B}$.

Let $b \in A_1^+ \setminus A_{in}$. Since $A_{in} \cap A_{sa}$ is dense in $A_{sa}$, there exists a sequence $\{b_m\}$ in $A_{in} \cap A_{sa}$ such that $b_m \to b$ as $m \to \infty$. Put $c_m = \frac{1}{|b_m|}b_m$. Then $c_m \to \frac{1}{|b|}b = b$ as $m \to \infty$. Put $a_m = \sqrt{c_m}c_m$. Then $a_m \to b$ as $m \to \infty$ and $a_m \in A_1^+ \cap A_{in}$. Thus there exists a sequence $\{a_m\}$ in $A_1^+ \cap A_{in}$ such that $a_m \to b$ as $m \to \infty$, and by the continuity of $T$

$$\lim_{m \to \infty} T(a_m x) = T(\lim_{m \to \infty} a_m x) = T(bx)$$

for all $x \in \mathcal{A}\mathcal{B}$. By (3.1),

$$\|T(a_m x) - bT(x)\| = \|a_m T(x) - bT(x)\| \to \|bT(x) - bT(x)\| = 0$$

as $m \to \infty$. By (3.2) and (3.3),

$$\|T(bx) - bT(x)\| \leq \|T(bx) - T(a_m x)\| + \|T(a_m x) - bT(x)\|$$

$$\to 0 \quad \text{as} \quad m \to \infty$$

(3.4)

for all $x \in \mathcal{A}\mathcal{B}$. By (3.1) and (3.4), $T(ax) = aT(x)$ for all $a \in A_1^+ \cup \{i\}$ and all $x \in \mathcal{A}\mathcal{B}$.

The rest of the proof is similar to the proof of Theorem 3.2. □

**Theorem 3.4.** Let $q$ be not an integer. Let $f : \mathcal{A}\mathcal{B} \to \mathcal{A}\mathcal{C}$ be a mapping with $f(0) = 0$ such that

$$\|D_a f(x_1, \ldots, x_n)\| \leq \varphi(x_1, \ldots, x_n)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x_1, \ldots, x_n \in \mathcal{A}\mathcal{B}$. If $f(\lambda x)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $x \in \mathcal{A}\mathcal{B}$, then there exists a unique $\mathcal{A}$–linear mapping $T : \mathcal{A}\mathcal{B} \to \mathcal{A}\mathcal{C}$ satisfying $(2.ii)$.

**Proof.** Put $a = 1 \in A_1^+$. By Theorem 3.1, there exists a unique $\mathbb{R}$–linear mapping $T : \mathcal{A}\mathcal{B} \to \mathcal{A}\mathcal{C}$ satisfying $(2.ii)$.

The rest of the proof is similar to the proof of Theorem 3.2. □

**Theorem 3.5.** Let $A$ be a unital $C^*$–algebra of real rank 0, and $q$ not an integer. Let $f : \mathcal{A}\mathcal{B} \to \mathcal{A}\mathcal{C}$ be a mapping with $f(0) = 0$ such that

$$\|D_a f(x_1, \ldots, x_n)\| \leq \varphi(x_1, \ldots, x_n)$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x_1, \ldots, x_n \in \mathcal{A}\mathcal{B}$. Assume that $f(ax)$ is continuous in $a \in A_1 \cup \mathbb{R}$ for each fixed $x \in \mathcal{A}\mathcal{B}$, and that the sequence $\{q^{-j}f(q^jx)\}$ converges uniformly on $A_1$ for each fixed $x \in \mathcal{A}\mathcal{B}$. Then there exists a unique $\mathcal{A}$–linear mapping $T : \mathcal{A}\mathcal{B} \to \mathcal{A}\mathcal{C}$ satisfying $(2.ii)$. 
Proof. By the same reasoning as the proof of Theorem 3.2, there exists a unique $\mathbb{R}$–linear mapping $T : A\mathcal{B} \rightarrow A\mathcal{C}$ satisfying (2.ii), and

$$T(ax) = aT(x)$$

for all $a \in (A^+ \cap A^n) \cup \{i\}$ and all $x \in A\mathcal{B}$. By the continuity of $f$ and the uniform convergence, one can show that $T(ax)$ is continuous in $a \in A_1$ for each $x \in A\mathcal{B}$.

The rest of the proof is similar to the proof of Theorem 3.3.

Theorem 3.6. Let $f : A\mathcal{B} \rightarrow A\mathcal{C}$ be a mapping with $f(0) = 0$ such that

$$\|D_1f(x_1, \cdots, x_n)\| \leq \varphi(x_1, \cdots, x_n)$$

for all $a \in A_1^+ \cup \{i\} \cup \mathbb{R}$ and all $x_1, \cdots, x_n \in A\mathcal{B}$. Then there exists a unique $A$–linear mapping $T : A\mathcal{B} \rightarrow A\mathcal{C}$ satisfying (2.ii).

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique additive mapping $T : A\mathcal{B} \rightarrow A\mathcal{C}$ satisfying (2.ii), and

$$T(ax) = aT(x)$$

for all $a \in A_1^+ \cup \{i\} \cup \mathbb{R}$ and all $x \in A\mathcal{B}$. So the mapping $T : A\mathcal{B} \rightarrow A\mathcal{C}$ is $\mathbb{R}$–linear, and satisfies

$$T(ax) = aT(x)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x \in A\mathcal{B}$.

The rest of the proof is the same as the proof of Theorem 3.2.

Similarly, for the case that $\varphi : A\mathcal{B}^n \rightarrow [0, \infty)$ is a function such that

$$\sum_{j=0}^{\infty} q^j \varphi(q^{-j}x_1, \cdots, q^{-j}x_n) < \infty$$

for all $x_1, \cdots, x_n \in A\mathcal{B}$, one can obtain similar results to the theorems given above.


In this section, let $A$ and $B$ be Banach algebras with norms $\| \cdot \|$ and $\| \cdot \|$, respectively.


We prove the Hyers–Ulam–Rassias stability of algebra homomorphisms between Banach algebras associated with the functional equation (A).

Theorem 4.1. Let $A$ and $B$ be real Banach algebras, and $q$ not an integer. Let $f : A \rightarrow B$ be a mapping with $f(0) = 0$ for which there exists a function $\psi : A \times A \rightarrow [0, \infty)$ such that

$$\tilde{\psi}(x, y) := \sum_{j=0}^{\infty} q^{-j} \psi(q^j x, y) < \infty,$$  \hspace{1cm} (4.i)

$$\|D_1f(x_1, \cdots, x_n)\| \leq \varphi(x_1, \cdots, x_n),$$  \hspace{1cm} (4.ii)

$$\|f(x \cdot y) - f(x)f(y)\| \leq \psi(x, y)$$  \hspace{1cm} (4.iii)
for all $x, y, x_1, \ldots, x_n \in \mathcal{A}$. If $f(\lambda x)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then there exists a unique algebra homomorphism $T : \mathcal{A} \rightarrow \mathcal{B}$ satisfying (2.ii). Further, if $\mathcal{A}$ and $\mathcal{B}$ are unital, then $f$ itself is an algebra homomorphism.

**Proof.** Under the assumption (2.i) and (4.ii), in Theorem 3.1, we showed that there exists a unique $\mathbb{R}$-linear mapping $T : \mathcal{A} \rightarrow \mathcal{B}$ satisfying (2.ii). The $\mathbb{R}$-linear mapping $T : \mathcal{A} \rightarrow \mathcal{B}$ was given by

$$T(x) = \lim_{j \rightarrow \infty} q^{-j} f(q^j x)$$

for all $x \in \mathcal{A}$. Let

$$R(x, y) = f(x \cdot y) - f(x)f(y)$$

for all $x, y \in \mathcal{A}$. By (4.i), we get

$$\lim_{j \rightarrow \infty} q^{-j} R(q^j x, y) = 0$$

for all $x, y \in \mathcal{A}$. So

$$T(x \cdot y) = \lim_{j \rightarrow \infty} q^{-j}$$

and

$$f(q^j(x \cdot y)) = \lim_{j \rightarrow \infty} q^{-j} f((q^j x)y)$$

$$= \lim_{j \rightarrow \infty} q^{-j} (f(q^j x)f(y) + R(q^j x, y)) = T(x)f(y)$$

(4.2)

for all $x, y \in \mathcal{A}$. Thus

$$T(x)f(q^j y) = T((q^j y)x) = T(q^j x)f(y) = q^j T(x)f(y)$$

for all $x, y \in \mathcal{A}$. Hence

$$T(x)q^{-j} f(q^j y) = T(x)f(y)$$

(4.3)

for all $x, y \in \mathcal{A}$. Taking the limit in (4.2) as $j \rightarrow \infty$, we obtain

$$T(x)f(y) = T(x)f(y)$$

for all $x, y \in \mathcal{A}$. Therefore,

$$T(x \cdot y) = T(x)T(y)$$

for all $x, y \in \mathcal{A}$. So $T : \mathcal{A} \rightarrow \mathcal{B}$ is an algebra homomorphism.

Now assume that $\mathcal{A}$ and $\mathcal{B}$ are unital. By (4.2),

$$T(y) = T(1 \cdot y) = T(1)f(y) = f(y)$$

for all $y \in \mathcal{A}$. So $f : \mathcal{A} \rightarrow \mathcal{B}$ is an algebra homomorphism, as desired. \qed

**Theorem 4.2.** Let $\mathcal{A}$ and $\mathcal{B}$ be complex Banach algebras. Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a mapping with $f(0) = 0$ for which there exists a function $\psi : \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ satisfying (4.ii) and (4.iii) such that

$$\|D_\lambda f(x_1, \ldots, x_n)\| \leq \psi(x_1, \ldots, x_n)$$

(4.iv)

for all $\lambda \in \mathcal{T}^1$ and all $x_1, \ldots, x_n \in \mathcal{A}$. Then there exists a unique algebra homomorphism $T : \mathcal{A} \rightarrow \mathcal{B}$ satisfying (2.ii). Further, if $\mathcal{A}$ and $\mathcal{B}$ are unital, then $f$ itself is an algebra homomorphism.

**Proof.** Under the assumption (2.i) and (4.iv), in Corollary 2.2, we showed that there exists a unique $\mathbb{C}$-linear mapping $T : \mathcal{A} \rightarrow \mathcal{B}$ satisfying (2.ii).

The rest of the proof is the same as the proof of Theorem 4.2. \qed
Theorem 4.3. Let $\mathcal{A}$ and $\mathcal{B}$ be complex Banach $*$-algebras. Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping with $f(0) = 0$ for which there exists a function $\psi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ satisfying (4.i) and (4.iii) such that
\[\|D_\lambda f(x_1, \cdots, x_n)\| \leq \varphi(x_1, \cdots, x_n)\] (4.iv)
\[\|f(x^*) - f(x)^*\| \leq \varphi(x, \cdots, x)\] (4.v)
for all $\lambda \in \mathbb{T}$ and all $x, x_1, \cdots, x_n \in \mathcal{A}$. Then there exists a unique $*$-algebra homomorphism $T : \mathcal{A} \to \mathcal{B}$ satisfying (2.ii). Further, if $\mathcal{A}$ and $\mathcal{B}$ are unital, then $f$ itself is a $*$-algebra homomorphism.

Proof. By the same reasoning as the proof of Theorem 4.2, there exists a unique $C$–linear mapping $T : \mathcal{A} \to \mathcal{B}$ satisfying (2.ii).

Now
\[q^{-j}\|f(q^jx^*) - f(q^jx)^*\| \leq q^{-j}\varphi(q^jx, \cdots, q^jx)\]
for all $x \in \mathcal{A}$. Thus
\[q^{-j}\|f(q^jx^*) - f(q^jx)^*\| \to 0\]
as $n \to \infty$ for all $x \in \mathcal{A}$. Hence
\[T(x^*) = \lim_{j \to \infty} q^{-j}f(q^jx^*) = \lim_{j \to \infty} q^{-j}f(q^jx)^* = T(x)^*\]
for all $x \in \mathcal{A}$.

The rest of the proof is the same as the proof of Theorem 4.2. \qed

Similarly, for the case that $\varphi : \mathcal{A}^n \to [0, \infty)$ and $\psi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ are functions such that
\[\sum_{j=0}^{\infty} q^j\varphi(q^{-j}x_1, \cdots, q^{-j}x_n) < \infty,\]
\[\sum_{j=0}^{\infty} q^j\psi(q^{-j}x, y) < \infty\]
for all $x, y, x_1, \cdots, x_n \in \mathcal{A}$, one can obtain similar results to the theorems given above.

References


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