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GENERALIZED POPOVICIU FUNCTIONAL EQUATIONS IN BANACH MODULES OVER A C^*–ALGEBRA AND APPROXIMATE ALGEBRA HOMOMORPHISMS

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Abstract. We prove the Hyers–Ulam–Rassias stability of generalized Popoviciu functional equations in Banach modules over a unital C^*–algebra. It is applied to show the stability of Banach algebra homomorphisms between Banach algebras associated with generalized Popoviciu functional equations in Banach algebras.

1. Introduction

Let E_1 and E_2 be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Consider $f : E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\varepsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \ \leq \ \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$. Th.M. Rassias [**11**] showed that there exists a unique $\mathbb{R}$–linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \ \leq \ \frac{2\varepsilon}{2-2^p}\|x\|^p$$

for all $x \in E_1$.

Throughout this paper, let A be a unital C^*–algebra with norm $|\cdot|$, $\mathcal{U}(A)$ the unitary group of A, A_{in} the set of invertible elements in A, A_{sa} the set of self-adjoint elements in A, $A_1 = \{a \in A \mid |a| = 1\}$, and A_1^+ the set of positive elements in A_1. Let ${}_A\mathcal{B}$ and ${}_A\mathcal{C}$ be left Banach A–modules with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Let n and k be integers with $2 \leq k \leq n-1$.

Recently, T. Trif [**19**] generalized the Popoviciu functional equation

$$3f\left(\frac{x+y+z}{3}\right)+f(x)+f(y)+f(z) \ = \ 2\left(f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right).$$

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Lemma A. [**19**, Theorem 2.1] Let V and W be vector spaces. A mapping $f : V \to W$ with $f(0) = 0$ satisfies the functional equation

$$n \cdot {}_{n-2}C_{k-2} f\left(\frac{x_1 + \cdots + x_n}{n}\right) + {}_{n-2}C_{k-1} \sum_{i=1}^{n} f(x_i) = k \cdot \sum_{1 \le i_1 < \cdots < i_k \le n} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right) \quad \text{(A)}$$

for all $x_1, \cdots, x_n \in V$ if and only if the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x + y) = f(x) + f(y)$ for all $x, y \in V$.

The following is useful to prove the stability of the functional equation (A).

Lemma B. [**10**, Theorem 1] Let $a \in A$ and $|a| < 1 - \frac{2}{m}$ for some integer m greater than 2. Then there are m elements $u_1, \cdots, u_m \in \mathcal{U}(A)$ such that $ma = u_1 + \cdots + u_m$.

The main purpose of this paper is to prove the Hyers–Ulam–Rassias stability of the functional equation (A) in Banach modules over a unital C^*–algebra, and to prove the Hyers–Ulam–Rassias stability of algebra homomorphisms between Banach algebras associated with the functional equation (A).

2. Stability of Generalized Popoviciu Functional Equations in Banach Modules over a C^*–Algebra Associated with its Unitary Group

We are going to prove the Hyers–Ulam–Rassias stability of the functional equation (A) in Banach modules over a unital C^*–algebra associated with its unitary group.

For a given mapping $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ and a given $a \in A$, we set

$$D_a f(x_1, \cdots, x_n) := n \cdot {}_{n-2}C_{k-2}\, a f\left(\frac{x_1 + \cdots + x_n}{n}\right) + {}_{n-2}C_{k-1} \sum_{i=1}^{n} a f(x_i) - k \cdot \sum_{1 \le i_1 < \cdots < i_k \le n} f\left(\frac{a x_{i_1} + \cdots + a x_{i_k}}{k}\right)$$

for all $x_1, \cdots, x_n \in {}_A\mathcal{B}$.

Theorem 2.1. *Let $q = \frac{k(n-1)}{n-k}$ and $r = -\frac{k}{n-k}$. Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_A\mathcal{B}^n \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} q^{-j} \varphi(q^j x_1, \cdots, q^j x_n) < \infty \|D_u f(x_1, \cdots, x_n)\| \le \varphi(x_1, \cdots, x_n) \quad \text{(2.i)}$$

for all $u \in \mathcal{U}(A)$ and all $x_1, \cdots, x_n \in {}_A\mathcal{B}$. Then there exists a unique A–linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ such that

$$\|f(x) - T(x)\| \le \frac{1}{k \cdot {}_{n-1}C_{k-1}} \widetilde{\varphi}(qx, rx, \cdots, rx) \quad \text{(2.ii)}$$

for all $x \in {}_A\mathcal{B}$.

Proof. Put $u = 1 \in \mathcal{U}(A)$. By [**19**, Theorem 3.1], there exists a unique additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (2.*ii*). The additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ was defined by

$$T(x) \ = \ \lim_{j\to\infty} q^{-j} f(q^j x)$$

for all $x \in {}_A\mathcal{B}$.

By the assumption, for each $u \in \mathcal{U}(A)$,

$$q^{-j}\|D_u f(q^j x, \cdots, q^j x)\| \ \le \ q^{-j}\varphi(q^j x, \cdots, q^j x)$$

for all $x \in {}_A\mathcal{B}$, and

$$q^{-j}\|D_u f(q^j x, \cdots, q^j x)\| \ \to \ 0$$

as $n \to \infty$ for all $x \in {}_A\mathcal{B}$. So

$$D_u T(x, \cdots, x) = \lim_{j\to\infty} q^{-j} D_u f(q^j x, \cdots, q^j x) = 0$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. Hence

$$D_u T(x, \cdots, x) = n \cdot {}_{n-2}C_{k-2} uT(x) + n \cdot {}_{n-2}C_{k-1} uT(x) - k \cdot {}_nC_k T(ux) = 0$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. So

$$uT(x) \ = \ T(ux)$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$.

Now let $a \in A$ ($a \neq 0$) and M an integer greater than $4|a|$. Then

$$|\frac{a}{M}| = \frac{1}{M}|a| < \frac{|a|}{4|a|} = \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3} \ .$$

By Lemma B, there exist three elements $u_1, u_2, u_3 \in \mathcal{U}(A)$ such that $3\frac{a}{M} = u_1 + u_2 + u_3$. And $T(x) = T\left(3 \cdot \frac{1}{3}x\right) = 3T\left(\frac{1}{3}x\right)$ for all $x \in {}_A\mathcal{B}$. So $T\left(\frac{1}{3}x\right) = \frac{1}{3}T(x)$ for all $x \in {}_A\mathcal{B}$. Thus

$$\begin{aligned} T(ax) \ &= \ T\left(\frac{M}{3} \cdot 3\frac{a}{M}x\right) = M \cdot T\left(\frac{1}{3} \cdot 3\frac{a}{M}x\right) = \frac{M}{3}T\left(3\frac{a}{M}x\right) \\ &= \ \frac{M}{3}T(u_1x + u_2x + u_3x) = \frac{M}{3}\big(T(u_1x) + T(u_2x) + T(u_3x)\big) \\ &= \ \frac{M}{3}(u_1 + u_2 + u_3)T(x) = \frac{M}{3} \cdot 3\frac{a}{M}\,T(x) \\ &= \ aT(x) \end{aligned}$$

for all $x \in {}_A\mathcal{B}$. Obviously, $T(0x) = 0T(x)$ for all $x \in {}_A\mathcal{B}$. Hence

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in {}_A\mathcal{B}$. So the unique additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is an A–linear mapping, as desired. □

Applying the unital C^*–algebra $\mathbb{C}$ to Theorem 2.1, one can obtain the following.

Corollary 2.2. *Let E_1 and E_2 be complex Banach spaces with norms $\| \cdot \|$ and $\| \cdot \|$, respectively. Let $q = \frac{k(n-1)}{n-k}$ and $r = -\frac{k}{n-k}$. Let $f : E_1 \to E_2$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : E_1^n \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} q^{-j} \varphi(q^j x_1, \cdots, q^j x_n)$$

$$< \infty \| D_\lambda f(x_1, \cdots, x_n) \| \le \varphi(x_1, \cdots, x_n)$$

for all $\lambda \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and all $x_1, \cdots, x_n \in E_1$. Then there exists a unique $\mathbb{C}$–linear mapping $T : E_1 \to E_2$ such that

$$\| f(x) - T(x) \| \le \frac{1}{k \cdot {}_{n-1}C_{k-1}} \widetilde{\varphi}(qx, rx, \cdots, rx)$$

for all $x \in E_1$.

Theorem 2.3. *Let $q = \frac{k(n-1)}{n-k}$ and $r = -\frac{k}{n-k}$. Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a continuous mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_A\mathcal{B}^n \to [0, \infty)$ satisfying (2.i) such that*

$$\| D_u f(x_1, \cdots, x_n) \| \le \varphi(x_1, \cdots, x_n)$$

for all $u \in \mathcal{U}(A)$ and all $x_1, \cdots, x_n \in {}_A\mathcal{B}$. If the sequence $\{q^{-j} f(q^j x)\}$ converges uniformly on ${}_A\mathcal{B}$, then there exists a unique continuous A–linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (2.ii).

Proof. Put $u = 1 \in \mathcal{U}(A)$. By Theorem 2.1, there exists a unique A–linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (2.*ii*). By the continuity of f, the uniform convergence and the definition of T, the A–linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is continuous, as desired. □

Theorem 2.4. *Let $q = \frac{k(n-1)}{n-k}$ and $r = -\frac{1}{n-k}$. Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_A\mathcal{B}^n \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x_1, \cdots, x_n) : = \sum_{j=0}^{\infty} q^j \varphi(q^{-j} x_1, \cdots, q^{-j} x_n) < \infty i$$

$$\| D_u f(x_1, \cdots, x_n) \| \le \varphi(x_1, \cdots, x_n) \tag{2.iii}$$

for all $u \in \mathcal{U}(A)$ and all $x_1, \cdots, x_n \in {}_A\mathcal{B}$. Then there exists a unique A–linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ such that

$$\| f(x) - T(x) \| \le \frac{1}{{}_{n-2}C_{k-1}} \widetilde{\varphi}(x, rx, \cdots, rx) \tag{2.iv}$$

for all $x \in {}_A\mathcal{B}$.

Proof. Put $u = 1 \in \mathcal{U}(A)$. By [**19**, Theorem 3.2], there exists a unique additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (2.*iv*). The additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ was defined by

$$T(x) = \lim_{j \to \infty} q^j f(q^{-j} x)$$

for all $x \in {}_A\mathcal{B}$.

By the assumption, for each $u \in \mathcal{U}(A)$,

$$q^j \| D_u f(q^{-j} x, \cdots, q^{-j} x) \| \le q^j \varphi(q^{-j} x, \cdots, q^{-j} x)$$

for all $x \in {}_A\mathcal{B}$, and

$$q^j \|D_u f(q^{-j}x, \cdots, q^{-j}x)\| \ \to \ 0$$

as $n \to \infty$ for all $x \in {}_A\mathcal{B}$. So

$$D_u T(x, \cdots, x) = \lim_{j\to\infty} q^j D_u f(q^{-j}x, \cdots, q^{-j}x) = 0$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. Hence

$$D_u T(x, \cdots, x) = n \cdot {}_{n-2}C_{k-2} uT(x) + n \cdot {}_{n-2}C_{k-1} uT(x) - k \cdot {}_n C_k T(ux) = 0$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. So

$$uT(x) \ = \ T(ux)$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$.

The rest of the proof is the same as the proof of Theorem 2.1. □

3. Stability of Generalized Popoviciu Functional Equations in Banach Modules over a C^*–Algebra

Given a locally compact abelian group G and a multiplier ω on G, one can associate to them the twisted group C^*–algebra $C^*(G, \omega)$. $C^*(\mathbb{Z}^m, \omega)$ is said to be a *noncommutative torus of rank* m and denoted by A_ω. The multiplier ω determines a subgroup S_ω of G, called its *symmetry group*, and the multiplier is called *totally skew* if the symmetry group S_ω is trivial. And A_ω is called *completely irrational* if ω is totally skew (see [**2**]). It was shown in [**2**] that if G is a locally compact abelian group and ω is a totally skew multiplier on G, then $C^*(G, \omega)$ is a simple C^*–algebra. It was shown in [**3**, Theorem 1.5] that if A_ω is a simple noncommutative torus then A_ω has real rank 0, where "*real rank* 0" means that the set of invertible self-adjoint elements is dense in the set of self–adjoint elements (see [**3**], [**6**]).

From now on, assume that $\varphi : {}_A\mathcal{B}^n \to [0, \infty)$ is a function satisfying (2.i), and that $q = \frac{k(n-1)}{n-k}$ and $r = -\frac{k}{n-k}$.

We are going to prove the Hyers–Ulam–Rassias stability of the functional equation (A) in Banach modules over a unital C^*–algebra.

Theorem 3.1. *Let q be not an integer. Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ such that*

$$\|D_1 f(x_1, \cdots, x_n)\| \ \le \ \varphi(x_1, \cdots, x_n)$$

for all $x_1, \cdots, x_n \in {}_A\mathcal{B}$. Then there exists a unique additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (2.ii). Further, if $f(\lambda x)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{B}$, then the additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is $\mathbb{R}$–linear.

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (2.ii).

Assume that $f(\lambda x)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{B}$. By the assumption, q is a rational number which is not an integer. The additive mapping T given above is similar to the additive mapping T given in the proof of [**11**, Theorem]. By the same reasoning as the proof of [**11**, Theorem], the additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is $\mathbb{R}$–linear. □

Theorem 3.2. *Let q be not an integer. Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a continuous mapping with $f(0) = 0$ such that*

$$\|D_a f(x_1, \cdots, x_n)\| \ \leq \ \varphi(x_1, \cdots, x_n)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x_1, \cdots, x_n \in {}_A\mathcal{B}$. If the sequence $\{q^{-j} f(q^j x)\}$ converges uniformly on ${}_A\mathcal{B}$, then there exists a unique continuous A–linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying $(2.ii)$.

Proof. Put $a = 1 \in A_1^+$. By Theorem 3.1, there exists a unique $\mathbb{R}$–linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying $(2.ii)$. By the continuity of f and the uniform convergence, the $\mathbb{R}$–linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is continuous.

By the same reasoning as the proof of Theorem 2.1,

$$T(ax) \ = \ aT(x)$$

for all $a \in A_1^+ \cup \{i\}$.

For any element $a \in A$, $a = \frac{a+a^*}{2} + i\frac{a-a^*}{2i}$, and $\frac{a+a^*}{2}$ and $\frac{a-a^*}{2i}$ are self–adjoint elements, furthermore, $a = \left(\frac{a+a^*}{2}\right)^+ - \left(\frac{a+a^*}{2}\right)^- + i\left(\frac{a-a^*}{2i}\right)^+ - i\left(\frac{a-a^*}{2i}\right)^-$, where $\left(\frac{a+a^*}{2}\right)^+$, $\left(\frac{a+a^*}{2}\right)^-$, $\left(\frac{a-a^*}{2i}\right)^+$, and $\left(\frac{a-a^*}{2i}\right)^-$ are positive elements (see [**4**, Lemma 38.8]). So

$$\begin{aligned}
T(ax) &= T\left(\left(\frac{a+a^*}{2}\right)^+ x - \left(\frac{a+a^*}{2}\right)^- x + i\left(\frac{a-a^*}{2i}\right)^+ x - i\left(\frac{a-a^*}{2i}\right)^- x\right) \\
&= \left(\frac{a+a^*}{2}\right)^+ T(x) + \left(\frac{a+a^*}{2}\right)^- T(-x) + \left(\frac{a-a^*}{2i}\right)^+ T(ix) \\
&\quad + \left(\frac{a-a^*}{2i}\right)^- T(-ix) \\
&= \left(\frac{a+a^*}{2}\right)^+ T(x) - \left(\frac{a+a^*}{2}\right)^- T(x) + i\left(\frac{a-a^*}{2i}\right)^+ T(x) \\
&\quad - i\left(\frac{a-a^*}{2i}\right)^- T(x) \\
&= \left(\left(\frac{a+a^*}{2}\right)^+ - \left(\frac{a+a^*}{2}\right)^- + i\left(\frac{a-a^*}{2i}\right)^+ - i\left(\frac{a-a^*}{2i}\right)^-\right) T(x) \\
&= aT(x)
\end{aligned}$$

for all $a \in A$ and all $x \in {}_A\mathcal{B}$. Hence

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in {}_A\mathcal{B}$, as desired. □

Theorem 3.3. *Let A be a unital C^*–algebra of real rank 0, and q not an integer. Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a continuous mapping with $f(0) = 0$ such that*

$$\|D_a f(x_1, \cdots, x_n)\| \ \leq \ \varphi(x_1, \cdots, x_n)$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ *and all* $x_1, \cdots, x_n \in {}_A\mathcal{B}$. *If the sequence* $\{q^{-j} f(q^j x)\}$ *converges uniformly on* ${}_A\mathcal{B}$*, then there exists a unique continuous* A*–linear mapping* $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ *satisfying* $(2.ii)$.

Proof. By the same reasoning as the proof of Theorem 3.2, there exists a unique continuous $\mathbb{R}$–linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying $(2.ii)$, and

$$T(ax) = aT(x) \tag{3.1}$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x \in {}_A\mathcal{B}$.

Let $b \in A_1^+ \setminus A_{in}$. Since $A_{in} \cap A_{sa}$ is dense in A_{sa}, there exists a sequence $\{b_m\}$ in $A_{in} \cap A_{sa}$ such that $b_m \to b$ as $m \to \infty$. Put $c_m = \frac{1}{|b_m|} b_m$. Then $c_m \to \frac{1}{|b|} b = b$ as $m \to \infty$. Put $a_m = \sqrt{c_m^* c_m}$. Then $a_m \to b$ as $m \to \infty$ and $a_m \in A_1^+ \cap A_{in}$. Thus there exists a sequence $\{a_m\}$ in $A_1^+ \cap A_{in}$ such that $a_m \to b$ as $m \to \infty$, and by the continuity of T

$$\lim_{m\to\infty} T(a_m x) = T(\lim_{m\to\infty} a_m x) = T(bx) \tag{3.2}$$

for all $x \in {}_A\mathcal{B}$. By (3.1),

$$\|T(a_m x) - bT(x)\| = \|a_m T(x) - bT(x)\| \to \|bT(x) - bT(x)\| = 0 \tag{3.3}$$

as $m \to \infty$. By (3.2) and (3.3),

$$\begin{aligned} \|T(bx) - bT(x)\| &\le \|T(bx) - T(a_m x)\| + \|T(a_m x) - bT(x)\| \\ &\to 0 \quad \text{as} \quad m \to \infty \end{aligned} \tag{3.4}$$

for all $x \in {}_A\mathcal{B}$. By (3.1) and (3.4), $T(ax) = aT(x)$ for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{B}$.

The rest of the proof is similar to the proof of Theorem 3.2. □

Theorem 3.4. *Let* q *be not an integer. Let* $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ *be a mapping with* $f(0) = 0$ *such that*

$$\|D_a f(x_1, \cdots, x_n)\| \le \varphi(x_1, \cdots, x_n)$$

for all $a \in A_1^+ \cup \{i\}$ *and all* $x_1, \cdots, x_n \in {}_A\mathcal{B}$. *If* $f(\lambda x)$ *is continuous in* $\lambda \in \mathbb{R}$ *for each fixed* $x \in {}_A\mathcal{B}$*, then there exists a unique* A*–linear mapping* $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ *satisfying* $(2.ii)$.

Proof. Put $a = 1 \in A_1^+$. By Theorem 3.1, there exists a unique $\mathbb{R}$–linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying $(2.ii)$.

The rest of the proof is similar to the proof of Theorem 3.2. □

Theorem 3.5. *Let* A *be a unital* C^**–algebra of real rank* 0*, and* q *not an integer. Let* $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ *be a mapping with* $f(0) = 0$ *such that*

$$\|D_a f(x_1, \cdots, x_n)\| \le \varphi(x_1, \cdots, x_n)$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ *and all* $x_1, \cdots, x_n \in {}_A\mathcal{B}$. *Assume that* $f(ax)$ *is continuous in* $a \in A_1 \cup \mathbb{R}$ *for each fixed* $x \in {}_A\mathcal{B}$*, and that the sequence* $\{q^{-j} f(q^j x)\}$ *converges uniformly on* A_1 *for each fixed* $x \in {}_A\mathcal{B}$. *Then there exists a unique* A*–linear mapping* $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ *satisfying* $(2.ii)$.

Proof. By the same reasoning as the proof of Theorem 3.2, there exists a unique $\mathbb{R}$–linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (2.*ii*), and

$$T(ax) \;=\; aT(x)$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x \in {}_A\mathcal{B}$. By the continuity of f and the uniform convergence, one can show that $T(ax)$ is continuous in $a \in A_1$ for each $x \in {}_A\mathcal{B}$.

The rest of the proof is similar to the proof of Theorem 3.3. □

Theorem 3.6. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ such that*

$$\|D_a f(x_1, \cdots, x_n)\| \;\le\; \varphi(x_1, \cdots, x_n)$$

for all $a \in A_1^+ \cup \{i\} \cup \mathbb{R}$ and all $x_1, \cdots, x_n \in {}_A\mathcal{B}$. Then there exists a unique A–linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (2.*ii*).

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (2.*ii*), and

$$T(ax) \;=\; aT(x)$$

for all $a \in A_1^+ \cup \{i\} \cup \mathbb{R}$ and all $x \in {}_A\mathcal{B}$. So the mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is $\mathbb{R}$–linear, and satisfies

$$T(ax) \;=\; aT(x)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{B}$.

The rest of the proof is the same as the proof of Theorem 3.2. □

Similarly, for the case that $\varphi : {}_A\mathcal{B}^n \to [0, \infty)$ is a function such that

$$\sum_{j=0}^{\infty} q^j \varphi(q^{-j}x_1, \cdots, q^{-j}x_n) \;<\; \infty$$

for all $x_1, \cdots, x_n \in {}_A\mathcal{B}$, one can obtain similar results to the theorems given above.

4. Stability of Generalized Popoviciu Functional Equations in Banach Algebras and Approximate Algebra Homomorphisms Associated with (A)

In this section, let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with norms $\|\cdot\|$ and $\|\cdot\|$, respectively.

D.G. Bourgin [**5**] proved the stability of ring homomorphisms between Banach algebras. In [**1**], R. Badora generalized the Bourgin's result.

We prove the Hyers–Ulam–Rassias stability of algebra homomorphisms between Banach algebras associated with the functional equation (A).

Theorem 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be real Banach algebras, and q not an integer. Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping with $f(0) = 0$ for which there exists a function $\psi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that*

$$\widetilde{\psi}(x, y) := \sum_{j=0}^{\infty} q^{-j} \psi(q^j x, y) \;<\; \infty, \tag{4.i}$$

$$\|D_1 f(x_1, \cdots, x_n)\| \;\le\; \varphi(x_1, \cdots, x_n), \tag{4.ii}$$

$$\|f(x \cdot y) - f(x)f(y)\| \;\le\; \psi(x, y) \tag{4.iii}$$

for all $x, y, x_1, \cdots, x_n \in \mathcal{A}$. If $f(\lambda x)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then there exists a unique algebra homomorphism $T : \mathcal{A} \to \mathcal{B}$ satisfying (2.ii). Further, if $\mathcal{A}$ and $\mathcal{B}$ are unital, then f itself is an algebra homomorphism.

Proof. Under the assumption (2.i) and (4.ii), in Theorem 3.1, we showed that there exists a unique $\mathbb{R}$–linear mapping $T : \mathcal{A} \to \mathcal{B}$ satisfying (2.ii). The $\mathbb{R}$–linear mapping $T : \mathcal{A} \to \mathcal{B}$ was given by

$$T(x) \;=\; \lim_{j\to\infty} q^{-j} f(q^j x)$$

for all $x \in \mathcal{A}$. Let

$$R(x, y) \;=\; f(x \cdot y) - f(x)f(y)$$

for all $x, y \in \mathcal{A}$. By (4.i), we get

$$\lim_{j\to\infty} q^{-j} R(q^j x, y) \;=\; 0$$

for all $x, y \in \mathcal{A}$. So

$$T(x \cdot y) \;=\; \lim_{j\to\infty} q^{-j} \tag{4.1}$$

$$\begin{aligned} f\big(q^j(x \cdot y)\big) &= \lim_{j\to\infty} q^{-j} f\big((q^j x)y\big) \\ &= \lim_{j\to\infty} q^{-j}\big(f(q^j x)f(y) + R(q^j x, y)\big) = T(x)f(y) \end{aligned} \tag{4.2}$$

for all $x, y \in \mathcal{A}$. Thus

$$T(x)f(q^j y) = T\big(x(q^j y)\big) = T\big((q^j x)y\big) = T(q^j x)f(y) = q^j T(x)f(y)$$

for all $x, y \in \mathcal{A}$. Hence

$$T(x)q^{-j} f(q^j y) \;=\; T(x)f(y) \tag{4.3}$$

for all $x, y \in \mathcal{A}$. Taking the limit in (4.2) as $j \to \infty$, we obtain

$$T(x)T(y) \;=\; T(x)f(y)$$

for all $x, y \in \mathcal{A}$. Therefore,

$$T(x \cdot y) \;=\; T(x)T(y)$$

for all $x, y \in \mathcal{A}$. So $T : \mathcal{A} \to \mathcal{B}$ is an algebra homomorphism.

Now assume that $\mathcal{A}$ and $\mathcal{B}$ are unital. By (4.2),

$$T(y) \;=\; T(1 \cdot y) = T(1)f(y) = f(y)$$

for all $y \in \mathcal{A}$. So $f : \mathcal{A} \to \mathcal{B}$ is an algebra homomorphism, as desired. □

Theorem 4.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be complex Banach algebras. Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping with $f(0) = 0$ for which there exists a function $\psi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ satisfying (4.i) and (4.iii) such that*

$$\|D_\lambda f(x_1, \cdots, x_n)\| \;\le\; \varphi(x_1, \cdots, x_n) \tag{4.iv}$$

for all $\lambda \in \mathbb{T}^1$ and all $x_1, \cdots, x_n \in \mathcal{A}$. Then there exists a unique algebra homomorphism $T : \mathcal{A} \to \mathcal{B}$ satisfying (2.ii). Further, if $\mathcal{A}$ and $\mathcal{B}$ are unital, then f itself is an algebra homomorphism.

Proof. Under the assumption (2.i) and (4.iv), in Corollary 2.2, we showed that there exists a unique $\mathbb{C}$–linear mapping $T : \mathcal{A} \to \mathcal{B}$ satisfying (2.ii).

The rest of the proof is the same as the proof of Theorem 4.2. □

Theorem 4.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be complex Banach $*$–algebras. Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping with $f(0) = 0$ for which there exists a function $\psi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ satisfying* (4.*i*) *and* (4.*iii*) *such that*

$$\|D_\lambda f(x_1, \cdots, x_n)\| \leq \varphi(x_1, \cdots, x_n) \tag{4.iv}$$

$$\|f(x^*) - f(x)^*\| \leq \varphi(x, \cdots, x) \tag{4.v}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, x_1, \cdots, x_n \in \mathcal{A}$. Then there exists a unique $*$–algebra homomorphism $T : \mathcal{A} \to \mathcal{B}$ satisfying (2.*ii*)*. Further, if $\mathcal{A}$ and $\mathcal{B}$ are unital, then f itself is a $*$–algebra homomorphism.*

Proof. By the same reasoning as the proof of Theorem 4.2, there exists a unique $\mathbb{C}$–linear mapping $T : \mathcal{A} \to \mathcal{B}$ satisfying (2.*ii*).

Now

$$q^{-j}\|f(q^j x^*) - f(q^j x)^*\| \leq q^{-j}\varphi(q^j x, \cdots, q^j x)$$

for all $x \in \mathcal{A}$. Thus

$$q^{-j}\|f(q^j x^*) - f(q^j x)^*\| \to 0$$

as $n \to \infty$ for all $x \in \mathcal{A}$. Hence

$$T(x^*) = \lim_{j\to\infty} q^{-j} f(q^j x^*) = \lim_{j\to\infty} q^{-j} f(q^j x)^* = T(x)^*$$

for all $x \in \mathcal{A}$.

The rest of the proof is the same as the proof of Theorem 4.2. □

Similarly, for the case that $\varphi : \mathcal{A}^n \to [0, \infty)$ and $\psi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ are functions such that

$$\sum_{j=0}^{\infty} q^j \varphi(q^{-j} x_1, \cdots, q^{-j} x_n) < \infty,$$

$$\sum_{j=0}^{\infty} q^j \psi(q^{-j} x, y) < \infty$$

for all $x, y, x_1, \cdots, x_n \in \mathcal{A}$, one can obtain similar results to the theorems given above.

References

1. R. Badora, *On approximate ring homomorphisms*, J. Math. Anal. Appl. **276** (2002), 589–597.
2. L. Baggett and A. Kleppner, *Multiplier representations of abelian groups*, J. Funct. Anal. **14** (1973), 299–324.
3. B. Blackadar, A. Kumjian and M. Rørdam, *Approximately central matrix units and the structure of noncommutative tori*, K–Theory **6** (1992), 267–284.
4. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer–Verlag, New York, Heidelberg and Berlin, 1973.
5. D.G. Bourgin, *Approximately isometric and multiplicative transformations on continuous function rings*, Duke Math. J. **16** (1949), 385–397.
6. L. Brown and G. Pedersen, *C^*–algebras of real rank zero*, J. Funct. Anal. **99** (1991), 131–149.

7. D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Berlin, Basel and Boston, 1998.
8. G. Isac and Th.M. Rassias, *On the Hyers–Ulam stability of ψ–additive mappings*, J. Approx. Theory **72** (1993), 131–137.
9. B.E. Johnson, *An approximately multiplicative map between Banach algebras*, J. London Math. Soc. **37** (1988), 294–316.
10. R. Kadison and G. Pedersen, *Means and convex combinations of unitary operators*, Math. Scand. **57** (1985), 249–266.
11. Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
12. Th.M. Rassias, *On a modified Hyers–Ulam sequence*, J. Math. Anal. Appl. **158** (1991), 106–113.
13. Th.M. Rassias, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. **251** (2000), 264–284.
14. Th.M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62** (2000), 123–130.
15. Th.M. Rassias and P. Šemrl, *On the behavior of mappings which does not satisfy Hyers–Ulam stability*, Proc. Amer. Math. Soc. **114** (1992), 989–993.
16. Th.M. Rassias and P. Šemrl, *On the Hyers–Ulam stability of linear mappings*, J. Math. Anal. Appl. **173** (1993), 325–338.
17. H. Schröder, *K–Theory for Real C^*–Algebras and Applications*, Pitman Research Notes in Mathematics, Longman, Essex, vol. 290, 1993.
18. T. Trif, *Hyers–Ulam-Rassias stability of a Jensen type functional equation*, J. Math. Anal. Appl. **250** (2000), 579–588.
19. T. Trif, *On the stability of a functional equation deriving from an inequality of Popoviciu for convex functions*, J. Math. Anal. Appl. **272** (2002), 604–616.

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