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SCALARIZATION OF THE NORMAL FRÉCHET REGULARITY OF SET–VALUED MAPPINGS

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Abstract. Let M be a set–valued mapping defined between two Banach spaces E and F. Several important aspects of behavior of M can be characterized in terms of the distance function to images Δ_M defined by $\Delta_M(x,y) := d\big(y, M(x)\big)$ for all $(x,y) \in E \times F$. In this paper, we use this function to scalarize the Fréchet normal regularity of set–valued mappings. The Fréchet subdifferential regularity of Δ_M is also investigated for points outside of the graph. An application to the metric regularity for set–valued mappings is given.

1. Introduction

Throughout the paper, we use standard notations with some special symbols introduced where they are defined. We will let M denote a set–valued mapping defined between two normed vector spaces E and F. If not specified, $\|\cdot\|$ and $\mathbb{B}_Z$ will denote respectively, the norm and the closed unit ball centered at the origin of a given normed vector space Z. The graph gph M (resp. the effective domain dom M) of $M : E \Longrightarrow F$ is the set gph $M := \{(x,y) \in E \times F : y \in M(x)\}$ (resp. dom $M := \{x \in E : M(x) \neq \varnothing\}$). The set–valued mapping M is said to be closed (resp. closed–valued), if its graph is (resp. its images are) closed. For a set–valued mapping $M : E \Longrightarrow F$, we associate the distance function Δ_M defined on $E \times F$ by $\Delta(x,y) := d\big(y, M(x)\big)$, where $d(\,\cdot\,, S)$ denotes the usual distance function to the set $S \subset F$, i.e., $d(y,S) := \inf\{\|y-s\| : s \in S\}$, with the convention $d(y,S) = +\infty$ when S is empty. For a set–valued mapping M and a given real number $r \geq 0$, we define the r*–enlargement* set–valued mapping $M_r : E \Longrightarrow F$ by $M_r(x) := \{y \in F : \Delta_M(x,y) \leq r\}$. It is obvious that gph $M_r = \{(x,y) \in E \times F : \Delta_M(x,y) \leq r\}$ and that the 0–enlargement set–valued mapping M_0 coincides with M, whenever M is closed–valued.

Any set–valued mapping $M : E \Longrightarrow F$, can be identified, in someway, with its distance function Δ_M, and hence every property of M must correspond to a property of Δ_M and vice versa. In [**6**], the authors characterized the tangential regularity of a set–valued mapping M as the directional regularity of the scalar function Δ_M. Several other important aspects of the behavior of a set–valued mapping M have been characterized in terms of the scalar function Δ_M. One knows (see for example, Castaing and Valadier [**11**]) that under general assumptions the measurability

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of M is equivalent to the measurability of Δ_M. Rockafellar's result in [**37**] says that the pseudo–Lipschitzness of M at $(\overline{x}, \overline{y})$ in gph M is equivalent to the Lipschitzness of Δ_M around $(\overline{x}, \overline{y})$. These results can be seen as a scalarization of some properties of set–valued mappings. The reverse operation was studied in [**9**]. The authors established a "geomerization" of some scalar properties of set–valued mappings and provided some important applications. The most important of them was a geometric characterization of set–valued mappings M for which the scalar function Δ_M is directionally Lipschitz in the sense of Rockafellar [**36**]. Note that this scalar function has been successfully used by a number of authors including Clarke [**12, 13, 14**], Castaing and Valadier [**11**], Rockafellar [**37**], Thibault [**39**], Ioffe [**17**], and Bounkhel and Thibault [**6, 9, 10**]. Our primary goal of this paper is to scalarize, with the help of Δ_M, the normal Fréchet regularity of a set–valued mapping M and to give an application to metric regularity for set–valued mappings. Note that this notion, the normal Fréchet regularity, plays a crucial role in optimization and nonsmooth analysis. We refer the reader to [**5, 25, 26, 27, 28, 31, 32, 33**] for many results that make clear the importance of this notion. In this paper, we generalize in two ways some recent results proved by Bounkhel and Thibault [**8**] for subsets. First, we extend their results for set-valued mappings (moving subsets) and then we obtain their results as corollaries by taking the constant set–valued mapping. The second way is when a subset $S :=$ gph M is the graph of some set–valued mapping M, the results in [**8**] scalarize the normal Fréchet regularity of S with the help of the usual distance function to the graph $d_{\mathrm{gph}\, M}$, but our results here scalarize it with the help of the distance function to images Δ_M, which is more better than the scalarization with the distance function to the graph $d_{\mathrm{gph}\, M}$, because generally it is more convenient to handle with Δ_M. For example, when M is a usual mapping f, one has $\Delta_f(x, y) = \|y - f(x)\|$. The paper is organized as follows. In section two we recall some notations and preliminaries that are used in all the sequel of the paper. Section three is devoted to study the relationships between the Fréchet normal cone to the graph of a set–valued mapping and the Fréchet subdifferential of the distance function to images associated with this set–valued mapping. We establish that, when $(\overline{x}, \overline{y}) \in$ gph M, a relationship between the Fréchet subdifferential of Δ_M at $(\overline{x}, \overline{y})$ and the Fréchet normal cone of gph M at $(\overline{x}, \overline{y})$. This relationship allows us to relate the subdifferential Fréchet regularity of Δ_M at $(\overline{x}, \overline{y})$ to the normal Fréchet regularity of M at the same point. In section four, we assume that $(\overline{x}, \overline{y})$ is outside gph M and we put $r := \Delta_M(\overline{x}, \overline{y}) > 0$. We prove in this section a relationship between the Fréchet subdifferential of Δ_M at $(\overline{x}, \overline{y})$ and the Fréchet normal cone of gph M_r at the same point $(\overline{x}, \overline{y})$. We close this section by using this description, as in the first case, to relate the subdifferential Fréchet regularity of Δ_M at $(\overline{x}, \overline{y})$ to the normal Fréchet regularity of M_r at the same point $(\overline{x}, \overline{y})$. In the last section, we give an application to the metric regularity for set–valued mappings.

2. Preliminaries

Let f be an extended–real–valued function from a normed vector space E into $\mathbb{R} \cup \{-\infty, +\infty\}$ with $|f(\overline{x})| < +\infty$. We recall that the *Clarke* (resp. *Fréchet*) *subdifferential* of f at $\overline{x}$ is defined by (see [**35, 18**])

$$\partial^C f(\overline{x}) \;=\; \{x^* \in E^* : \langle x^*, h\rangle \le f^\uparrow(\overline{x}; h) \text{ for all } h \in E\},$$

where $f^\uparrow(\overline{x};\cdot)$ is *the generalized Rockafellar directional derivative* defined by

$$\begin{aligned} f^\uparrow(\overline{x};h) \;&=\; \limsup_{\substack{(x,\alpha)\downarrow_f \overline{x}\\ t\downarrow 0}} \inf_{h'\to h} t^{-1}\big[f(x+th')-\alpha\big] \\ &:=\; \sup_{H\in\mathcal{N}(h)} \Big[\limsup_{\substack{(x,\alpha)\downarrow_f \overline{x}\\ t\downarrow 0}} \Big(\inf_{h'\in H} t^{-1}\big[f(x+th')-\alpha\big]\Big)\Big], \end{aligned}$$

where $(x,\alpha)\downarrow_f \overline{x}$ means $(x,\alpha)\in \operatorname{epi} f := \{(z,\beta)\in E\times\mathbb{R}; f(z)\le\beta\}$ and $(x,\alpha)\longrightarrow \big(\overline{x}, f(\overline{x})\big)$ and $\mathcal{N}(h)$ denotes the filter of neighborhoods of h

$$\left(\text{resp.}\quad \partial^F f(x) \;=\; \left\{x^*\in X^* : \liminf_{x'\to x} \frac{f(x')-f(x)-\langle x^*, x'-x\rangle}{\|x'-x\|} \ge 0\right\}\right).$$

By convention, we set $\partial^F f(\overline{x}) = \partial^C f(\overline{x}) = \varnothing$, when $f(\overline{x})$ is not finite. Note that one always has $\partial^F f(\overline{x}) \subset \partial^C f(\overline{x})$.

Recall also that *the lower Hadamard directional derivative* of f at $\overline{x}$ is defined (see[**34**]) by

$$f^H(\overline{x};h) \;=\; \liminf_{\substack{h'\to h\\ t\downarrow 0}} t^{-1}\big[f(\overline{x}+th')-f(\overline{x})\big].$$

Note that if f is locally Lipschitz around $\overline{x}$, then $f^\uparrow(\overline{x};h)$ $\big($resp. $f^H(\overline{x};h)\big)$ coincides with the *Clarke* (resp. the *lower Dini*) *directional derivative* $f^0(\overline{x};\cdot)$ $\big($resp. $f^-(\overline{x};\cdot)\big)$ defined by

$$f^0(\overline{x};h) \;=\; \limsup_{\substack{x\to\overline{x}\\ t\downarrow 0}} t^{-1}\big[f(x+th)-f(x)\big].$$

$$\left(\textit{resp. } f^-(\overline{x};h) \;=\; \liminf_{t\downarrow 0} t^{-1}\big[f(\overline{x}+th)-f(\overline{x})\big]\right).$$

It is clear that one always has $f^H(\overline{x};\cdot) \le f^\uparrow(\overline{x};\cdot)$. Following Clarke [**12**] (see also [**5, 7, 8**]), one says that f is *directionally regular* at $\overline{x}$ if and only if equality holds in the last inequality, that is $f^H(\overline{x};\cdot) = f^\uparrow(\overline{x};\cdot)$.

Consider now another natural dual notion of regularity of functions in nonsmooth analysis that has been introduced and studied in [**7**]. A function f is said to be *Fréchet subdifferentially regular* at $\overline{x}$ whenever $\partial^F f(\overline{x})$ coincides with $\partial^C f(\overline{x})$. It is not difficult to check (see [**5, 7**]) that for any point $\overline{x}$ where $f(\overline{x})$ is finite and $\partial^C f(\overline{x}) \ne \varnothing$, f is directionally regular at $\overline{x}$ whenever it is Fréchet subdifferentially regular at $\overline{x}$. For a complete study concerning these notions, we refer the reader to [**5, 7, 8**].

Let S be a nonempty closed subset of E and $\overline{u}$ be a point in S. Let us recall that the *Fréchet normal cone* of S at $\overline{u}$ is defined by $N^F(S;\overline{u}) := \partial^F\psi_S(\overline{u})$, where ψ_S denotes the indicator function of S, i.e., $\psi_S(u) = 0$ if $u\in S$ and $+\infty$ otherwise.

It is not difficult to see that the Fréchet normal cone is also given by

$$N^F(S;\overline{u}) \;=\; \left\{x^*\in X^* : \limsup_{u\to^S \overline{u}} \left\langle x^*, \frac{u-\overline{u}}{\|u-\overline{u}\|}\right\rangle \le 0\right\},$$

where $u \longrightarrow^S \overline{u}$ means $u\longrightarrow\overline{u}$ and $u\in S$.

We are ready now to consider the following notion of normal regularity for sets.

Definition 2.1. A nonempty closed subset S of E is said to be *Fréchet normally regular* at a point $\overline{u} \in S$, if and only if $N^F(S;u) = N^C(S;u)$.

Now we recall some concepts for set–valued mappings. A set–valued mapping M will be said to be *Fréchet normally regular* at $(\overline{x},\overline{y}) \in \text{gph } M$, if its graph gph M is Fréchet normally regular at $(\overline{x},\overline{y})$. We can check that this definition is equivalent to the Fréchet coderivative regularity of M at $(\overline{x},\overline{y})$ in the sense that the Fréchet coderivative of M at $(\overline{x},\overline{y})$ coincides with the Clarke coderivative of M at $(\overline{x},\overline{y})$. Recall (see [**28**]) that the #–coderivative of M at some point $(\overline{x},\overline{y}) \in \text{gph } M$ (# = Fréchet, Clarke) is the set–valued mapping defined from F^* into E^* by

$$D^*_{\#}M(\overline{x},\overline{y})(y^*) \; = \; \{x^* \in E^* : (x^*,-y^*) \in N^{\#}(\text{gph } M;\overline{x},\overline{y})\}.$$

This concept of regularity have been used by Mordukhovich (see for instance [**25, 26**]) to establish equality forms, under other natural qualification conditions, in calculus rules for coderivatives of compositions, sums and intersections of set–valued mappings. We refer to [**25, 26, 27, 28, 31, 32, 33**] and the references therein for many results that make clear the importance of this notion.

Let us recall an important result on the Fréchet normal regularity for sets showed in [**8**]. For any nonempty closed subset of a reflexive Banach space E and any $\overline{u} \in S$, the Fréchet normal regularity of S at $\overline{u}$ coincides with the Mordukhovich regularity of S at $\overline{u}$. Recall that S is said to be Mordukhovich regular at $\overline{u}$ provided $N^F(S;\overline{u})$ coincides with the limiting Fréchet (or Mordukhovich) normal cone $N^{\mathrm{F.L}}(S;\overline{u})$ defined (see [**24**]), when E is an Asplund space, by $N^{\mathrm{F.L}}(S;\overline{u}) :=^s \limsup_{u\to\overline{u}} N^F(S;u)$, that means $u^* \in N^{F.L}(S;\overline{u})$, if and only if there exist sequences $u_n \to^S \overline{u}$ and $u_n^* \to^{w^*} u^*$ in E^* with $u_n^* \in N^F(S;u_n)$, where $\to^{w^*}$ means the w^*–convergence.

We close this section by recalling the following result showed in [**4**].

Proposition 2.2 ([**4**]). *Let $M : E \Longrightarrow F$ be a set–valued mapping and $(\overline{x},\overline{y}) \notin$ gph M. Put $r := \Delta_M(\overline{x},\overline{y}) > 0$. Assume that Δ_M is directionally regular at $(\overline{x},\overline{y})$ and $\mathrm{Proj}_{M(\overline{x})}(\overline{y}) \neq \varnothing$. Then one has*

$$\partial^C\Delta_M(\overline{x},\overline{y}) \subset N^C(\text{gph } M_r;\overline{x},\overline{y}) \;\cap\; (E^* \times \{y^* \in F^* : \|y^*\| = 1\}).$$

Here $\mathrm{Proj}_{M(\overline{x})}(\overline{y}) := \{y \in F : \Delta_M(\overline{x},\overline{y}) = \|\overline{y} - y\|\}$.

Note that When F is a finite dimensional space, $\mathrm{Proj}_{M(x)}(y)$ is always nonempty as long as $M(x)$ is nonempty and closed.

3. Scalarization of Fréchet Normal Regularity

Throughout this section $E \times F$ will be endowed by the sum norm, i.e., $\|(x,y)\| = \|x\| + \|y\|$. Our aim here is to give a scalarization of the Fréchet normal regularity of a large class of set-valued mappings, with the help of the scalar function Δ_M, that is, we show the equivalence between the Fréchet normal regularity of M at $(\overline{x},\overline{y}) \in \text{gph } M$ and the Fréchet subdifferential regularity of Δ_M at the same point. We begin with the definition of lower-pseudo–Lipschitz set–valued mappings.

Definition 3.1. Let M be any set-valued mapping defined between E and F and let $(\overline{x},\overline{y}) \in \text{gph } M$. We will say that M is *lower–pseudo-Lipschitz* at $(\overline{x},\overline{y})$ if there exist neighborhoods X of $\overline{x}$, Y of $\overline{y}$ and a constant $l \geq 0$ such that

$$M(\overline{x}) \cap Y \subset M(x) + l\|x - \overline{x}\|\mathbb{B}_F, \quad \text{for all} \quad x \in X.$$

Remark 3.2. It is obvious that M is lower–pseudo-Lipschitz at $(\overline{x}, \overline{y})$ whenever it is either pseudo–Lipschitz at the same point $(\overline{x}, \overline{y})$ in the sense of Aubin [**1**] or lower–Lipschitz at $\overline{x}$ in the sense of Denintion 4.2 in the next section. The converse does not hold. Indeed, the set valued mappings M_1 and M_2 defined as

$$M_1(x) \;=\; \begin{cases} \{0\}, & \text{if} \quad x = 0; \\ [-1, +1], & \text{otherwise,} \end{cases} \qquad M_2(x) \;=\; \begin{cases} [-1, +1], & \text{if} \quad x = 0; \\ |x| + 2, & \text{otherwise,} \end{cases}$$

are lower–pseudo–Lipschitz at $(0, 0)$ but M_1 is not pseudo–Lipschitz at $(0, 0)$ and M_2 is not lower–Lipschitz at 0.

We give the following lemma needed in the next theorem. Its proof follows the same ideas of the proof given by Rockafellar [**37**] to establish the important characterization of pseudo–Lipschitz set–valued mappings in terms of Δ_M.

Lemma 3.3. *Let $M : E \Longrightarrow F$ be any set–valued mapping and let $(\overline{x}, \overline{y}) \in \text{gph } M$. Then M is lower–pseudo–Lipschitz at $(\overline{x}, \overline{y})$, if and only if, there exist neighborhoods X of $\overline{x}$, Y of $\overline{y}$, and a constant $l \geq 0$ such that*

$$\Delta_M(x, y) \leq \Delta_M(\overline{x}, y) + l\|x - \overline{x}\|,$$

for all $y \in Y$ and all $x \in X \cap \text{dom} M$.

Now, we establish our first main result in this section.

Theorem 3.4. *Let E and F be two Banach spaces and let $M : E \Longrightarrow F$ be any set–valued mapping with $(\overline{x}, \overline{y}) \in \text{gph } M$. Then one has*

$$\partial^F \Delta_M(\overline{x}, \overline{y}) \subset N^F(\text{gph } M; \overline{x}, \overline{y}) \cap (E^* \times \mathbb{B}_{F^*}). \tag{3.1}$$

Furthermore, if M is lower–pseudo-Lipschitz at $(\overline{x}, \overline{y})$ with a constant $l \geq 0$, then one has

$$\begin{aligned} \partial^F \Delta_M(\overline{x}, \overline{y}) \;&=\; N^F(\text{gph} M; \overline{x}, \overline{y}) \cap (l\mathbb{B}_{E^*} \times \mathbb{B}_{F^*}) \\ &=\; N^F(\text{gph} M; \overline{x}, \overline{y}) \cap (E^* \times \mathbb{B}_{F^*}). \end{aligned} \tag{3.2}$$

Proof. We begin by proving (3.1). Fix $(x^*, y^*) \in \partial^F \Delta_M(\overline{x}, \overline{y})$. Then, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $(x, y) \in (\overline{x}, \overline{y}) + \delta\mathbb{B}_{E \times F}$

$$\langle (x^*, y^*), (x, y) - (\overline{x}, \overline{y}) \rangle \;\leq\; \Delta_M(x, y) - \Delta_M(\overline{x}, \overline{y}) + \varepsilon\|(x, y) - (\overline{x}, \overline{y})\|.$$

Hence,

$$\langle (x^*, y^*), (x, y) - (\overline{x}, \overline{y}) \rangle \;\leq\; \varepsilon\|(x, y) - (\overline{x}, \overline{y})\|,$$

for all $(x, y) \in \Big[(\overline{x}, \overline{y}) + \delta\mathbb{B}_{E \times F})\Big] \cap \text{gph } M$. This ensures that

$$(x^*, y^*) \;\in\; N^F(\text{gph } M; \overline{x}, \overline{y}).$$

On the other hand, as one always has $\partial^F \Delta_M(\overline{x}, \overline{y}) \subset \partial^C \Delta_M(\overline{x}, \overline{y}) \subset E^* \times \mathbb{B}_{F^*}$, one obtains $\partial^F \Delta_M(\overline{x}, \overline{y}) \subset E^* \times \mathbb{B}_{F^*}$. This completes the proof of the inclusion (3.1).

Now we will prove (3.2). Assume that M is lower–pseudo–Lipschitz at $(\overline{x}, \overline{y})$ with a constant $l \geq 0$. By Lemma 3.3 and Theorem 3.6 in [**6**], one can check that $\partial^C \Delta_M(\overline{x}, \overline{y}) \subset l\mathbb{B}_{E^*} \times \mathbb{B}_{F^*}$ and so by (3.1) one gets

$$\partial^F \Delta_M(\overline{x}, \overline{y}) \subset N^F(\text{gph } M; \overline{x}, \overline{y}) \cap (l\mathbb{B}_{E^*} \times \mathbb{B}_{F^*}).$$

To finish the proof of (3.2) we show the following inclusion

$$N^F(\text{gph } M; \overline{x}, \overline{y}) \cap (E^* \times \mathbb{B}_{F^*}) \subset \partial^F \Delta_M(\overline{x}, \overline{y}).$$

Fix (x^*, y^*) in $N^F(\text{gph } M; \overline{x}, \overline{y})$ with $\|y^*\| \leq 1$ and fix any $\varepsilon' > 0$. Fix also $0 < \varepsilon < \frac{\varepsilon'}{3+l}$. Then there exists $\delta > 0$ such that

$$\langle (x^*, y^*), (x, y) - (\overline{x}, \overline{y}) \rangle \;\leq\; \varepsilon \|(x, y) - (\overline{x}, \overline{y})\|, \tag{3.3}$$

for all $(x, y) \in \Big((\overline{x}, \overline{y}) + \delta \mathbb{B}_{E \times F}\Big) \cap \text{gph } M$. Now, as M is lower-pseudo–Lipschitz at $(\overline{x}, \overline{y})$, we can fix (by Lemma 3.3) a real number $\gamma := \min\Big\{1, \frac{\delta}{3+l}\Big\}$ for which one has

$$\Delta_M(x, y) \;\leq\; l\|x - \overline{x}\| + \|y - \overline{y}\|, \tag{3.4}$$

for all (x, y) in $(\overline{x}, \overline{y}) + \gamma \mathbb{B}_{E \times F}$. Fix also $0 < \eta < \frac{\varepsilon}{1+\varepsilon}$ and consider any $(x, y) \in (\overline{x}, \overline{y}) + \gamma \mathbb{B}_{E \times F}$. Choose $y' := y'(x, y)$ in $M(x)$ such that

$$\|y' - y\| \leq d\big(y, M(x)\big) + \eta \|(x, y) - (\overline{x}, \overline{y})\|. \tag{3.5}$$

Then,

$$\begin{aligned}
\|(x, y') - (\overline{x}, \overline{y})\| &\leq \|(x, y) - (\overline{x}, \overline{y})\| + \|y' - y\| \\
&\leq \gamma + d\big(y, M(x)\big) + \eta \|(x, y) - (\overline{x}, \overline{y})\| \\
&\leq (1 + \eta)\gamma + \|y - \overline{y}\| + l\|x - \overline{x}\| \qquad \big(\text{by (3.4)}\big) \\
&\leq (2 + l + \eta)\gamma \leq (3 + l)\gamma \leq \delta,
\end{aligned}$$

and hence $(x, y') \in \big((\overline{x}, \overline{y}) + \delta \mathbb{B}_{E \times F}\big) \cap \text{gph } M$. Thus by (3.3) one has

$$\begin{aligned}
\langle (x^*, y^*), (x, y) - (\overline{x}, \overline{y}) \rangle &= \langle (x^*, y^*), (x, y') - (\overline{x}, \overline{y}) \rangle + \langle y^*, y - y' \rangle \\
&\leq \varepsilon \|(x, y') - (\overline{x}, \overline{y})\| + \|y' - y\| \\
&\leq \varepsilon \|(x, y) - (\overline{x}, \overline{y})\| + \varepsilon \|y' - y\| + \|y' - y\|.
\end{aligned}$$

Therefore, one gets $\big($ by (3.5)$\big)$

$$\langle (x^*, y^*), (x, y) - (\overline{x}, \overline{y}) \rangle \;\leq\; \Delta_M(x, y) + (\eta + \varepsilon)\|(x, y) - (\overline{x}, \overline{y})\| + \varepsilon \|y' - y\|.$$

On the other hand, by (3.5) once again

$$\begin{aligned}
\|y' - y\| &\leq \Delta_M(x, y) + \eta \|(x, y) - (\overline{x}, \overline{y})\| \\
&\leq \|y - \overline{y}\| + l\|x - \overline{x}\| + \eta \|(x, y) - (\overline{x}, \overline{y})\| \qquad \big(\text{by (3.4)}\big) \\
&\leq (1 + l + \eta)\|(x, y) - (\overline{x}, \overline{y})\|.
\end{aligned}$$

Finally, one yields

$$\begin{aligned}
\langle (x^*, y^*), (x, y) - (\overline{x}, \overline{y}) \rangle &\leq \Delta_M(x, y) + \big(\eta(1 + \varepsilon) + \varepsilon(2 + l)\big)\|(x, y) - (\overline{x}, \overline{y})\| \\
&\leq \Delta_M(x, y) + \varepsilon(3 + l)\|(x, y) - (\overline{x}, \overline{y})\| \\
&\leq \Delta_M(x, y) - \Delta_M(\overline{x}, \overline{y}) + \varepsilon' \|(x, y) - (\overline{x}, \overline{y})\|.
\end{aligned}$$

This ensures that $(x^*, y^*) \in \partial^F \Delta_M(\overline{x}, \overline{y})$ and hence the proof is finished. $\square$

Now, we recall that a Banach space Z is *weakly compactly generated* (W.C.G.) provided there is a weakly compact set K such that $Z = \mathrm{cl}(\mathrm{span}\, K)$, where $\mathrm{span}\, K$ denotes the vector space generated by K. Clearly all reflexive Banach spaces and all separable Banach spaces are (W.C.G.). Recall also that S is said to be *compactly epi–Lipschitz* at $\overline{u} \in S$ (see [**2**]), if there exist a compact set $H \subset Z$ and real numbers $\gamma > 0$, $\varepsilon > 0$ such that

$$S \cap (\overline{u} + \varepsilon \mathbb{B}_Z) + t\mathbb{B}_Z \subset S - tH,$$

for all $t \in]0, \gamma[$. Note that, when Z is assumed to be an Asplund space, this notion is equivalent to the normal compactness with respect to the Fréchet normal cone, introduced by Loewen [**23**]. The direct implication has been proved first by Loewen [**23**] in reflexive Banach spaces. This implication has been extended by Mordukhovich and Shao [**31**] to Asplund spaces (see also Jourani and Thibault [**20**] for similar result with respect to the Ioffe approximate normal cone). Recently, Ioffe [**19**] proved that in Asplund spaces the reverse implication also holds, and hence a subset S is compactly epi–Lipschitz around a point $\overline{u}$ if and only if it is normally compact around $\overline{u}$ with respect to the Fréchet normal cone. We refer the reader to [**19, 20**] for other results in this vein about compactly epi–Lipschitz sets.

The following result by Mordukhovich and Shao [**31**] will be needed in the proof of Theorem 3.6.

Theorem 3.5 ([**31**]). *Let Z be an Asplund space and let S be a nonempty closed subset of Z with $\overline{u} \in S$. Suppose that S is compactly epi–Lipschitz at $\overline{u}$ and that Z is a* W.C.G. *space. Then $N^{FL}(S; \overline{u})$ is weak–star closed.*

Theorem 3.6. *Let E and F be two Banach spaces and let $M : E \Longrightarrow F$ be any set–valued mapping defined from E into F with $(\overline{x}, \overline{y}) \in \mathrm{gph}\ M$.*

(1) *If M is lower–pseudo–Lipschitz at $(\overline{x}, \overline{y})$ and Fréchet normally regular at $(\overline{x}, \overline{y})$, then the scalar function Δ_M is Fréchet subdifferentially regular at $(\overline{x}, \overline{y})$.*

(2) *If the scalar function Δ_M is Fréchet subdifferentially regular at $(\overline{x}, \overline{y})$ and either the spaces E and F are reflexive or* gph M *is compactly epi–Lipschitz at $(\overline{x}, \overline{y})$ and the product space $E \times F$ is* W.C.G *Asplund space, then the set–valued mapping M is Fréchet normally regular at $(\overline{x}, \overline{y})$.*

Proof. (1) Assume that M is lower–pseudo–Lipschitz and Fréchet normally regular at $(\overline{x}, \overline{y})$. Then by Corollary 2.1 in [**4**] and Theorem 3.4 one obtains

$$\begin{aligned} \partial^C \Delta_M(\overline{x}, \overline{y}) &\subset N^C(\mathrm{gph}\ M; \overline{x}, \overline{y}) \cap (E^* \times \mathbb{B}_{F^*}) \\ &= N^F(\mathrm{gph}\ M; \overline{x}, \overline{y}) \cap (E^* \times \mathbb{B}_{F^*}) \\ &= \partial^F \Delta_M(\overline{x}, \overline{y}), \end{aligned}$$

which ensures the Fréchet subdifferential regularity of Δ_M at $(\overline{x}, \overline{y})$.

(2) Now, we prove (2). Assume first that the scalar function Δ_M is Fréchet subdifferentially regular at $(\overline{x}, \overline{y})$ and that the spaces E and F are reflexive. The definition of Fréchet subdifferential regularity ensures that $\partial^F \Delta_M(\overline{x}, \overline{y}) = \partial^C \Delta_M(\overline{x}, \overline{y})$ and so by Corollary 2.1 in [**4**] and Theorem 3.4 one gets

$$\begin{aligned} N^C(\mathrm{gph}\ M; \overline{x}, \overline{y}) &= \mathrm{cl}_{w^*}\big(\mathbb{R}_+ \partial^F \Delta_M(\overline{x}, \overline{y})\big) \\ &\subset \mathrm{cl}_{w^*}\big(N^F(\mathrm{gph}\ M; \overline{x}, \overline{y})\big). \end{aligned}$$

By Proposition 3.1 in [**8**], the cone $N^F(\text{gph } M; \overline{x}, \overline{y})$ is strongly closed in $E^* \times F^*$, and one knows that it is convex, and hence weak star closed in $E^* \times F^*$, because E and F are reflexive. Thus,

$$\begin{array}{rcl} N^C(\text{gph } M; \overline{x}, \overline{y}) & \subset & \text{cl}_{w^*}\big(N^F(\text{gph } M; \overline{x}, \overline{y})\big) \\ & = & N^F(\text{gph } M; \overline{x}, \overline{y}), \end{array}$$

which ensures the Fréchet normal regularity of M at $(\overline{x}, \overline{y})$.

Now we assume that gph M is compactly epi–Lipschitz at $(\overline{x}, \overline{y})$, that $E \times F$ is an Asplund W.C.G. space, and that Δ_M is Fréchet subdifferentially regular at $(\overline{x}, \overline{y})$. The definition of the Fréchet subdifferential regularity and the definition of the limiting Fréchet subdifferential ensure that

$$\partial^F \Delta_M(\overline{x}, \overline{y}) = \partial^{\text{F.L}} \Delta_M(\overline{x}, \overline{y}) = \partial^C \Delta_M(\overline{x}, \overline{y}). \tag{3.6}$$

By (see Thibault [**39**])

$$N^{\text{F.L}}(\text{gph } M; \overline{x}, \overline{y}) \;=\; \mathbb{R}_+ \partial^{\text{F.L}} \Delta_M(\overline{x}, \overline{y}), \tag{3.7}$$

and (see Corollary 2.1 in [**4**])

$$N^C(\text{gph } M; \overline{x}, \overline{y}) \;=\; \text{cl}_{w^*}\big(\mathbb{R}_+ \partial^C \Delta_M(\overline{x}, \overline{y})\big),$$

one gets

$$\begin{array}{rcl} N^C(\text{gph } M; \overline{x}, \overline{y}) & = & \text{cl}_{w^*}\big(\mathbb{R}_+ \partial^{\text{F.L}} \Delta_M(\overline{x}, \overline{y})\big) \\ & = & \text{cl}_{w^*}\big(N^{\text{F.L}}(\text{gph } M; \overline{x}, \overline{y})\big). \end{array} \tag{3.8}$$

Furthermore, as gph M is compactly epi–Lipschitz at $(\overline{x}, \overline{y})$ and $E \times F$ is an Asplund W.C.G. space, one has by Theorem 3.5

$$\text{cl}_{w^*}\big(N^{\text{F.L}}(\text{gph } M; \overline{x}, \overline{y})\big) \;=\; N^{\text{F.L}}(\text{gph } M; \overline{x}, \overline{y}). \tag{3.9}$$

Thus by (3.6), (3.7), (3.8), (3.9), and Theorem 3.4 we conclude that

$$\begin{array}{rcl} N^C(\text{gph } M; \overline{x}, \overline{y}) & = & N^{\text{F.L}}(\text{gph } M; \overline{x}, \overline{y}) \\ & = & \mathbb{R}_+ \partial^{\text{F.L}} \Delta_M(\overline{x}, \overline{y}) \\ & = & \mathbb{R}_+ \partial^F \Delta_M(\overline{x}, \overline{y}) \\ & \subset & N^F(\text{gph } M; \overline{x}, \overline{y}), \end{array}$$

which completes the proof of the theorem. □

Before giving other characterizations of the Fréchet normal regularity, we will need some other results. So, we start with the following theorem.

Theorem 3.7. *Let X be a reflexive Banach space, $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be l.s.c. on X with $\overline{x} \in \text{dom } f$ and let $s(\overline{x}; \cdot)$ be the support function of $\partial^F f(\overline{x})$, that is*

$$s(\overline{x}; h) \;:=\; \sup\{\langle y^*, h\rangle; y^* \in \partial^F f(\overline{x})\}.$$

Suppose that f is directionally Lipschitz at $\overline{x}$. Then the following assertions are equivalent:

(i) *f is Fréchet subdifferentially regular at $\overline{x}$;*

(ii) *$\partial^F f$ is topologically closed at $\overline{x}$ and* $\text{dom } f^{\uparrow}(\overline{x}; \cdot) = \text{dom } s(\overline{x}; \cdot)$*;*

(ii) *$\partial^F f$ is sequentially closed at $\overline{x}$ and* $\text{dom } f^{\uparrow}(\overline{x}; \cdot) = \text{dom } s(\overline{x}; \cdot)$.

Recall that f is said to be directionally Lipschitz at a point $\overline{x} \in \operatorname{dom} f$, if and only if its epigraph is epi–Lipschitz at $\big(\overline{x}, f(\overline{x})\big)$ in the sense of Rockafellar [**36**]. Recall also that (see for instance [**4, 7**]) the Fréchet subdifferential $\partial^F f$ is said to be *topologically closed* at a point $\overline{x} \in \operatorname{dom} f$, if for every net $(x_j, x_j^*)_{j \in J}$ in $\partial^F f$ such that $x_j^* \longrightarrow^{w^*} x^*$ and $x_j \longrightarrow^f \overline{x}$ one has $(\overline{x}, x^*) \in \partial^F f$, where $\longrightarrow^{w^*}$ denotes the w^*–convergence in X^* and $(y, y^*) \in \partial^F f$ means that $y^* \in \partial^F f(y)$. When the set J is replaced by $\mathbb{N}$, we say that $\partial^F f$ is *sequentially closed* at $\overline{x}$.

Proof of Theorem 3.7. Observe that, as X is reflexive Banach space, then (by Proposition 3.2 in [**8**]) $\partial^F f(\overline{x})$ is weak star closed. Therefore, the arguments (with minor modifications) used in the proof of Theorem 5.1 in [**7**] give the conclusion of the theorem. □

Note that in [**7**], the authors showed the equivalence between (i) and (ii) when E is assumed to be an Asplund space. The equivalence of later conditions with (iii) is an open problem in the case of Asplund space. It cannot be deduced directly from Theorem 5.1 in [**7**], because the Fréchet subdifferential is not necessarily weak star closed in Asplund spaces.

Now we are in position to establish the following important characterizations of the Fréchet normal regularity for set–valued mappings.

Theorem 3.8. *Let E and F be two reflexive Banach spaces and let $M : E \Longrightarrow F$ be a pseudo–Lipschitz set–valued mapping defined from E into F with $(\overline{x}, \overline{y}) \in \operatorname{gph} M$. Then the following assertions are equivalent :*

(1) *M is Fréchet normally regular at $(\overline{x}, \overline{y})$;*

(2) *M is Mordukhovich regular at $(\overline{x}, \overline{y})$;*

(3) *Δ_M is Fréchet subdifferentially regular at $(\overline{x}, \overline{y})$;*

(4) *$\partial^F \Delta_M$ is topologically closed at $(\overline{x}, \overline{y})$;*

(5) *$\partial^F \Delta_M$ is sequentially closed at $(\overline{x}, \overline{y})$.*

Proof. The equivalence between (1) and (2) is proved in Theorem 3.4 in [**7**]. The other ones follow from Theorems 3.6 and 3.7. □

Corollary 3.9. *Let $M : \mathbb{R}^p \Longrightarrow \mathbb{R}^q$ be a pseudo–Lipschitz set–valued mapping at $(\overline{x}, \overline{y}) \in \operatorname{gph} M$. Then (1)–(5) are equivalent to the following assertions:*

(6) *M is tangentially regular at $(\overline{x}, \overline{y})$;*

(7) *Δ_M is directionally regular at $(\overline{x}, \overline{y})$;*

(8) *$d_{\operatorname{gph} M}$ is directionally regular at $(\overline{x}, \overline{y})$.*

4. Characterization of the Fréchet Subdifferential Regularity of Δ_M at Points Outside the Graph

For any set–valued mapping $M : E \Longrightarrow F$, the Fréchet normal regularity of M at a point $(\overline{x}, \overline{y})$ outside the graph cannot be considered because of the definition of the Fréchet normal cone which makes no sense in this case. However, the Fréchet subdifferential regularity of Δ_M at $(\overline{x}, \overline{y})$ can be considered for each $(\overline{x}, \overline{y}) \in \operatorname{dom} M \times F$ $\big($even when $(\overline{x}, \overline{y}) \notin \operatorname{gph} M\big)$. Therefore, our purpose in this section is to give an answer to the following question: *Is it possible to establish* *a geometric characterization for the subdifferential Fréchet regularity of the scalar function* Δ_M *at* $(\overline{x}, \overline{y}) \notin \text{gph } M$? In other words, what is the property of M which corresponds to the Fréchet subdifferential regularity of Δ_M at $(\overline{x}, \overline{y}) \notin \text{gph } M$?

We begin with the following important Lemma, which is in the line of Lemma 3.1 in [**8**].

Lemma 4.1. *Let* $M : E \Longrightarrow F$ *be a closed–valued mapping and let* $r > 0$*. Then for all* $(x, y) \notin \text{gph } M_r$ *one has*

$$\Delta_{M_r}(x, y) \;=\; \Delta_M(x, y) - r. \tag{4.1}$$

Proof. Fix any $(x, y) \notin \text{gph } M_r$. Consider any $y' \in M_r(x)$, that is, $\Delta_M(x, y') \leq r$, and consider also any $\varepsilon > 0$. We may choose some $y'_\varepsilon \in M(x)$ satisfying

$$\|y' - y'_\varepsilon\| \leq \Delta_M(x, y') + \varepsilon \leq \varepsilon + r.$$

Consequently,

$$\|y' - y\| \geq \|y'_\varepsilon - y\| - \|y'_\varepsilon - y'\| \;\geq\; \Delta_M(x, y) - r - \varepsilon.$$

As this inequality holds for all $y' \in M_r(x)$ and all $\varepsilon > 0$, we deduce

$$\Delta_{M_r}(x, y) \;\geq\; \Delta_M(x, y) - r.$$

Let us prove the reverse inequality. Fix any $y' \in M(x)$ and consider the continuous real–valued function h defined on $[0, +\infty[$ by $h(s) := \Delta_M\big(x, sy + (1 - s)y'\big)$. Observing that $h(0) = 0$ $\big($because $y' \in M(x)\big)$ and $h(1) > r > 0$ $\big($because $(x, y) \notin \text{gph } M_r\big)$, we may apply the classical intermidiate value theorem to get some $s_0 \in]0, 1[$ such that $h(s_0) = r$. Putting $z := s_0 y + (1 - s_0)y'$, we have $\Delta_M(x, z) = r$ and $\|y - y'\| = \|y - z\| + \|z - y'\|$. Therefore, because $y' \in M(x)$ we obtain

$$\|y - y'\| \geq \|y - z\| + \Delta_M(x, z) = \|y - z\| + r,$$

and as $z \in M_r(x)$, it follows that

$$\|y - y'\| \;\geq\; \Delta_{M_r}(x, y) + r.$$

This yields the inequality

$$\Delta_M(x, y) \;\geq\; \Delta_{M_r}(x, y) + r,$$

and hence the proof of the lemma is complete. □

Before establishing our main result in this section, we recall the following definition.

Definition 4.2. We will say that M is *upper–Lipschitz* (resp. *lower–Lipschitz*) at a given point $\overline{x} \in \text{dom } M$, if there exist a neighborhood X of $\overline{x}$ and a constant $l \geq 0$ such that

$$M(x) \subset M(\overline{x}) + l\|x - \overline{x}\|\mathbb{B}_F, \quad \text{for all} \quad x \in X.$$

(resp.

$$M(\overline{x}) \subset M(x) + l\|x - \overline{x}\|\mathbb{B}_F, \quad \text{for all} \quad x \in X \cap \text{dom } M.)$$

We say that M is *Lipschitz* at $\overline{x}$ if it is lower–Lipschitz and upper–Lipschitz at $\overline{x}$.

We state without proof the following lemma needed in the proof of the next theorem.

Lemma 4.3. *Let $M : E \Longrightarrow F$ be any set–valued mapping with $\overline{x} \in \operatorname{dom} M$. If M is upper–Lipschitz (resp. lower–Lipschitz) at $\overline{x}$ with a constant $l \geq 0$. Then there exists a neighborhood X of $\overline{x}$ such that for all $x \in X$ and $y \in F$ one has $\Delta_M(\overline{x}, y) \leq \Delta_M(x, y) + l\|x - \overline{x}\|$. (resp. $\Delta_M(x, y) \leq \Delta_M(\overline{x}, y) + l\|x - \overline{x}\|$.)*

The following theorem proves a relationship between the Fréchet subdifferential of Δ_M at a point $(\overline{x}, \overline{y}) \notin \operatorname{gph} M$ and the Fréchet normal cone of the graph of the r–enlargement set–valued mapping, with $r := \Delta_M(\overline{x}, \overline{y})$.

Theorem 4.4. *Let E and F be two Banach spaces and let $M : E \Longrightarrow F$ be any set–valued mapping with $(\overline{x}, \overline{y}) \notin \operatorname{gph} M$. Put $r := \Delta_M(\overline{x}, \overline{y})$. Then one has*

$$\partial^F \Delta_M(\overline{x}, \overline{y}) \subset N^F(\operatorname{gph} M_r; \overline{x}, \overline{y}) \cap (E^* \times \Omega). \tag{4.2}$$

(i) *If M is lower–Lipschitz at $\overline{x}$ with a constant $l \geq 0$, then (4.2) becomes*

$$\partial^F \Delta_M(\overline{x}, \overline{y}) \subset N^F(\operatorname{gph} M_r; \overline{x}, \overline{y}) \cap (l\mathbb{B}_{E^*} \times \Omega).$$

(ii) *If M is upper–Lipschitz at $\overline{x}$ with a constant $l \geq 0$, then there exists some equivalent norm on $E \times F$ such that, with respect to that norm, one has*

$$N^F(\operatorname{gph} M_r; \overline{x}, \overline{y}) \cap (l\mathbb{B}_{E^*} \times \Omega) \subset \partial^F \Delta_M(\overline{x}, \overline{y}).$$

Here $\Omega := \{y^* \in F^* : \|y^*\| = 1\}$

Proof. We begin by proving (4.2). Fix any $(x^*, y^*) \in \partial^F \Delta_M(\overline{x}, \overline{y})$ and any $\varepsilon > 0$. By the definition of the Fréchet subdifferential of Δ_M, there exists a real number $\delta > 0$ such that for all $(x, y) \in (\overline{x}, \overline{y}) + \delta\mathbb{B}_{E \times F}$ one has

$$\langle (x^*, y^*), (x - \overline{x}, y - \overline{y}) \rangle \leq \Delta_M(x, y) - \Delta_M(\overline{x}, \overline{y}) + \varepsilon\|(x - \overline{x}, y - \overline{y})\|. \tag{4.3}$$

By restricting the last inequality on the graph of M_r, one obtains

$$\langle (x^*, y^*), (x - \overline{x}, y - \overline{y}) \rangle \leq \varepsilon\|(x - \overline{x}, y - \overline{y})\|,$$

for all $(x, y) \in \big[(\overline{x}, \overline{y}) + \delta\mathbb{B}_{E \times F}\big] \cap \operatorname{gph} M_r$, which ensures that $(x^*, y^*) \in N^F (\operatorname{gph} M_r; \overline{x}, \overline{y})$.

Fix now any $b \in \mathbb{B}_E$ and taking $y := \overline{y} + \delta b$ and $x := \overline{x}$ in (4.3), one gets

$$\langle y^*, \delta b \rangle \leq \delta(1 + \varepsilon)\|b\| \leq \delta(1 + \varepsilon).$$

Since this holds for any $b \in \mathbb{B}_E$ and any $\varepsilon > 0$, one obtains $\|y^*\| \leq 1$.

To finish the proof of (4.2), we will show that $\|y^*\| \geq 1$. To this end, we fix $s := \min\left\{1, \frac{\delta}{1 + \Delta_M(\overline{x}, \overline{y})}, \varepsilon\right\}$ and choose $y_s \in M(\overline{x})$ such that

$$\|y_s - \overline{y}\| \leq d\big(\overline{y}, M(\overline{x})\big) + s^2. \tag{4.4}$$

Put $y' := \overline{y} + s(y_s - \overline{y})$. Since $\|y' - \overline{y}\| = s\|y_s - \overline{y}\| \leq sd\big(\overline{y}, M(\overline{x})\big) + s^3 \leq s\left(1 + d\big(\overline{y}, M(\overline{x})\big)\right) \leq \delta$, one gets $\big($by (4.3)$\big)$ with $y := y'$ and $x := \overline{x}$

$$\langle y^*, y' - \overline{y} \rangle \leq \Delta_M(\overline{x}, y') - \Delta_M(\overline{x}, \overline{y}) + \varepsilon\|y' - \overline{y}\|$$

and hence by (4.4)

$$\begin{aligned} \langle y^*, y' - \overline{y} \rangle &\leq \|y' - y_s\| - \|\overline{y} - y_s\| + s^2 + \varepsilon s\|\overline{y} - y_s\| \\ &= (1 - s)\|\overline{y} - y_s\| - \|\overline{y} - y_s\| + s^2 + \varepsilon s\|\overline{y} - y_s\| \\ &= -s\|\overline{y} - y_s\| + s^2 + \varepsilon s\|\overline{y} - y_s\|. \end{aligned}$$

Thus

$$\begin{aligned}\langle y^*, y_s - \overline{y}\rangle &\le -\|\overline{y} - y_s\| + s + \varepsilon\|\overline{y} - y_s\| \\ &\le -\|\overline{y} - y_s\| + \varepsilon(1 + \|\overline{y} - y_s\|),\end{aligned}$$

and hence

$$\begin{aligned}\frac{\langle y^*, \overline{y} - y_s\rangle}{\|\overline{y} - y_s\|} &\ge 1 - \varepsilon(1 + \frac{1}{\|\overline{y} - y_s\|}) \\ &\ge 1 - \varepsilon\left(1 + \frac{1}{\Delta_M(\overline{x}, \overline{y})}\right).\end{aligned}$$

By letting $\varepsilon \to 0^+$, one gets $\|y^*\| \ge 1$. This completes the proof of (4.2).

(i) Now, we assume that M is lower–Lipschitz at $\overline{x}$ with a constant $l \ge 0$. Fix any $(x^*, y^*) \in \partial^F \Delta_M(\overline{x}, \overline{y})$ and any $\varepsilon > 0$. By the definition of the Fréchet subdifferential of Δ_M and by Lemma 4.3, there exists a real number $\delta > 0$ such that (4.3) holds and

$$\Delta_M(x, y) \le \Delta_M(\overline{x}, y) + l\|x - \overline{x}\|,$$

for all $(x, y) \in (\overline{x}, \overline{y}) + \delta\mathbb{B}_{E\times F}$. Fix any $b \in \mathbb{B}_E$ and taking $y := \overline{y}$ and $x := \overline{x} + \delta b$ in the last inequality and in (4.3) one gets

$$\langle x^*, \delta b\rangle \le \delta(l + \varepsilon)\|b\| \le \delta(l + \varepsilon).$$

Since this holds for any $b \in \mathbb{B}_E$ and any $\varepsilon > 0$, one obtains $\|x^*\| \le l$. This completes the proof of (i).

(ii) Now, we assume that M is upper–Lipschitz at $\overline{x}$ with a constant $l \ge 0$. Let $|\!|\!|(\,\cdot\,,\cdot\,)|\!|\!|$ be the norm defined by $|\!|\!|(x, y)|\!|\!| := l\|x\| + \|y\|$ on $E \times F$ and let $|\!|\!|(\,\cdot\,,\cdot\,)|\!|\!|_*$ be the associated dual norm, $|\!|\!|(x^*, y^*)|\!|\!|_* := \sup\{\langle (x^*, y^*), (x, y)\rangle : |\!|\!|(x, y)|\!|\!| \le 1\}$. It is not difficult to see that $|\!|\!|(x^*, y^*)|\!|\!|_* = \max\{\frac{1}{l}\|x^*\|, \|y^*\|\}$. We will denote $\mathbb{B}$ and $\mathbb{B}_*$ their associated closed unit balls centered at the origins of $E \times F$ and $E^* \times F^*$ respectively. We will prove, with respect to these norms on the product spaces, the inclusion $N^F(\text{gph } M_r; \overline{x}, \overline{y}) \cap (l\mathbb{B}_{E^*} \times \Omega) \subset \partial^F \Delta_M(\overline{x}, \overline{y})$. Fix $(x^*, y^*) \in N^F$ $(\text{gph } M_r; \overline{x}, \overline{y})$, with $\|x^*\| \le l$ and $\|y^*\| = 1$ and fix any $\varepsilon > 0$ and any $\eta \in]0, 1[$. Observe that $(x^*, y^*) \in N^F(\text{gph } M_r; \overline{x}, \overline{y}) \cap \mathbb{B}_*$. So, by Theorem 3.1 in [**8**] one gets $(x^*, y^*) \in \partial^F d_{\text{gph } M_r}(\overline{x}, \overline{y})$ (where $d_{\text{gph } M_r}$ is the usual distance function associated with gph M_r and the norm $|\!|\!|(\,\cdot\,,\cdot\,)|\!|\!|$). Then there exists $\delta_1 > 0$ such that for all $(x, y) \in (\overline{x}, \overline{y}) + \delta_1\mathbb{B}$

$$\begin{aligned}\langle (x^*, y^*), (x - \overline{x}, y - \overline{y})\rangle &\le d_{\text{gph } M_r}(x, y) + \varepsilon|\!|\!|(x - \overline{x}, y - \overline{y})|\!|\!| \\ &\le \Delta_{M_r}(x, y) + \varepsilon|\!|\!|(x - \overline{x}, y - \overline{y})|\!|\!|.\end{aligned}$$

By Lemma 4.1, one gets for any $(x, y) \in \big[(\overline{x}, \overline{y}) + \delta_1\mathbb{B}\big] \setminus \text{gph } M_r$

$$\langle (x^*, y^*), (x - \overline{x}, y - \overline{y})\rangle \le \Delta_M(x, y) - \Delta_M(\overline{x}, \overline{y}) + \varepsilon|\!|\!|(x - \overline{x}, y - \overline{y})|\!|\!|. \tag{4.5}$$

On the other hand, as $(x^*, y^*) \in N^F(\text{gph } M_r; \overline{x}, \overline{y})$, there exists $\delta_2 > 0$ such that for all $(x, y) \in \big[(\overline{x}, \overline{y}) + \delta_2\mathbb{B}\big] \cap \text{gph } M_r$

$$\langle (x^*, y^*), (x - \overline{x}, y - \overline{y})\rangle \le \frac{\varepsilon}{2}|\!|\!|(x - \overline{x}, y - \overline{y})|\!|\!|. \tag{4.6}$$

Since $\|y^*\| = 1$, we can choose $u \in F$ with $\|u\| = 1$ such that $\langle y^*, u\rangle \geq 1 - \eta$. As M is upper–Lipschitz at $\overline{x}$, we can fix (by Lemma 4.3) $\delta_3 \in]0, \delta_2[$ for which one has

$$\Delta_M(\overline{x}, \overline{y}) \;\leq\; \Delta_M(x, y) + l\|x - \overline{x}\| + \|y - \overline{y}\|, \tag{4.7}$$

for all (x, y) in $(\overline{x}, \overline{y}) + \delta_3\mathbb{B}$.

Fix now $\delta_4 \in]0, \frac{\delta_3}{2}[$ and $(x, y) \in \big[(\overline{x}, \overline{y}) + \delta_4\mathbb{B}\big] \cap \text{gph } M_r$ and put $t := \Delta_M(\overline{x}, \overline{y}) - \Delta_M(x, y) \geq 0$. Then $(x, y + tu) \in \big[(\overline{x}, \overline{y}) + \delta_2\mathbb{B}\big] \cap \text{gph } M_r$, because $\Delta_M(x, y + tu) \leq \Delta_M(x, y) + t = \Delta_M(\overline{x}, \overline{y}) = r$, and

$$\begin{aligned}
|||(x, y + tu) - (\overline{x}, \overline{y})||| &= l\|x - \overline{x}\| + \|y + tu - \overline{y}\| \\
&\leq l\|x - \overline{x}\| + \|y - \overline{y}\| + t \\
&= |||(x, y) - (\overline{x}, \overline{y})||| + \Delta_M(\overline{x}, \overline{y}) - \Delta_M(x, y) \\
&\leq 2|||(x, y) - (\overline{x}, \overline{y})||| \qquad \big(\text{by } (4.7)\big) \\
&\leq 2\delta_4 \leq \delta_3 \leq \delta_2.
\end{aligned}$$

By (4.6) one gets

$$\begin{aligned}
\langle (x^*, y^*), (x, y + tu) - (\overline{x}, \overline{y})\rangle &\leq \frac{\varepsilon}{2}|||(x, y + tu) - (\overline{x}, \overline{y})||| \\
&\leq \frac{\varepsilon}{2}|||(x, y) - (\overline{x}, \overline{y})||| + t\frac{\varepsilon}{2} \\
&\leq \frac{\varepsilon}{2}|||(x, y) - (\overline{x}, \overline{y})||| + \frac{\varepsilon}{2}|||(x, y) - (\overline{x}, \overline{y})||| \\
&\leq \varepsilon|||(x, y) - (\overline{x}, \overline{y})|||,
\end{aligned}$$

and hence

$$\begin{aligned}
\langle (x^*, y^*), (x, y) - (\overline{x}, \overline{y})\rangle &= \langle (x^*, y^*), (x, y + tu) - (\overline{x}, \overline{y})\rangle - \langle y^*, tu\rangle \\
&\leq \varepsilon|||(x, y) - (\overline{x}, \overline{y})||| - t(1 - \eta) \\
&\leq \varepsilon|||(x, y) - (\overline{x}, \overline{y})||| + \big(\Delta_M(x, y) - \Delta_M(\overline{x}, \overline{y})\big)(1 - \eta).
\end{aligned}$$

As this holds for all $\eta > 0$, one deduces, for all $(x, y) \in \big[(\overline{x}, \overline{y}) + \delta_4\mathbb{B}\big] \cap \text{gph } M_r$

$$\langle (x^*, y^*), (x, y) - (\overline{x}, \overline{y})\rangle \;\leq\; \varepsilon|||(x, y) - (\overline{x}, \overline{y})||| + \big(\Delta_M(x, y) - \Delta_M(\overline{x}, \overline{y})\big). \tag{4.8}$$

According to (4.5) and (4.8), one obtains that for all $(x, y) \in (\overline{x}, \overline{y}) + \delta\mathbb{B}$ with $\delta := \min\{\delta_1, \delta_4\}$ one has

$$\langle (x^*, y^*), (x, y) - (\overline{x}, \overline{y})\rangle \;\leq\; \Delta_M(x, y) - \Delta_M(\overline{x}, \overline{y}) + \varepsilon|||(x, y) - (\overline{x}, \overline{y})|||.$$

So $(x^*, y^*) \in \partial^F\Delta_M(\overline{x}, \overline{y})$ and hence the proof of the theorem is complete. $\square$

As a direct consequence of the above theorem we have the following important result.

Theorem 4.5. *Let $M : E \Longrightarrow F$ be a set-valued mapping defined between two Banach spaces E and F with $(\overline{x}, \overline{y}) \notin \text{gph } M$. Put $r := \Delta_M(\overline{x}, \overline{y})$. Assume that M is Lipschitz at $\overline{x}$ with a constant $l \geq 0$. Then there exists some equivalent norm on $E \times F$ such that, with respect to that norm , the following equality holds*

$$\partial^F\Delta_M(\overline{x}, \overline{y}) = N^F(\text{gph } M_r; \overline{x}, \overline{y}) \cap (l\mathbb{B}_{E^*} \times \Omega).$$

Now, we are in position to establish our main result on the relationship between the Fréchet subdifferential regularity of Δ_M at a point $(\overline{x}, \overline{y})$ outside the graph gph M and the Fréchet normal regularity of the r–enlargement set-valued mapping M_r $\big($with $r := \Delta_M(\overline{x}, \overline{y})\big)$.

Theorem 4.6. *Let $M : E \Longrightarrow F$ be a set–valued mapping defined between two Banach spaces E and F with $(\overline{x}, \overline{y}) \notin \mathrm{gph}\ M$. Put $r := \Delta_M(\overline{x}, \overline{y})$. Assume that* $\mathrm{Proj}_{M(\overline{x})}(\overline{y}) \neq \varnothing$.

(i) *If the function Δ_M is Fréchet subdifferentially regular at $(\overline{x}, \overline{y})$ and E and F are reflexive Banach spaces, then M_r is Fréchet normally regular at $(\overline{x}, \overline{y})$.*

(ii) *Conversely, if M is Lipschitz at $\overline{x}$ and Δ_M is directionally regular at $(\overline{x}, \overline{y})$, then there exists some equivalent norm on $E \times F$ such that, Δ_M is Fréchet subdifferentially regular at $(\overline{x}, \overline{y})$ whenever the set–valued mapping M_r is Fréchet normally regular at $(\overline{x}, \overline{y})$ with respect to that norm.*

Proof. (i) Assume that Δ_M is Fréchet subdifferentially regular at $(\overline{x}, \overline{y})$, i.e., $\partial^F \Delta_M(\overline{x}, \overline{y}) = \partial^C \Delta_M(\overline{x}, \overline{y})$. Then, Δ_M is directionally regular at $(\overline{x}, \overline{y})$ and by Corollary 3.3 in [**4**], we get

$$\begin{aligned} N^C(\mathrm{gph}\ M_r; \overline{x}, \overline{y}) &= \mathrm{cl}_{w^*}\big(\mathbb{R}_+ \partial^C \Delta_M(\overline{x}, \overline{y})\big) \\ &= \mathrm{cl}_{w^*}\big(\mathbb{R}_+ \partial^F \Delta_M(\overline{x}, \overline{y})\big) \\ &\subset \mathrm{cl}_{w^*}\big(N^F(\mathrm{gph}\ M_r; \overline{x}, \overline{y})\big). \end{aligned}$$

By Proposition 3.1 in [**8**] , the cone $N^F(\mathrm{gph}\ M_r; \overline{x}, \overline{y})\big)$ is strongly closed in $E^* \times F^*$ and one knows that it is convex and hence it is weak star closed in $E^* \times F^*$ because E and F are reflexive. Thus $N^C(\mathrm{gph}\ M_r; \overline{x}, \overline{y}) \subset \mathrm{cl}_{w^*}\big(N^F(\mathrm{gph}\ M_r; \overline{x}, \overline{y})\big) = N^F(\mathrm{gph}\ M_r; \overline{x}, \overline{y})$. As the reverse inclusion always holds, we obtain the Fréchet normal regularity of M_r at $(\overline{x}, \overline{y})$.

(ii) Assume now that M is Lipschitz around $\overline{x}$ with a constant $l \geq 0$ and Δ_M is directionally regular at $(\overline{x}, \overline{y})$. By Proposition 2.2 one has

$$\partial^C \Delta_M(\overline{x}, \overline{y}) \subset N^C(\mathrm{gph}\ M_r; \overline{x}, \overline{y}) \cap \big(l\mathbb{B}_{E^*} \times \Omega\big). \tag{4.9}$$

Now, we assume that M_r is Fréchet normally regular at $(\overline{x}, \overline{y})$, with respect to the norm $\|\!|(\cdot\,,\,\cdot)|\!\|$ on $E \times F$ defined as in Theorem 4.4, that is, $N^F(\mathrm{gph}\ M_r; \overline{x}, \overline{y}) = N^C(\mathrm{gph}\ M_r; \overline{x}, \overline{y})$. Thus, according to (4.9) and Theorem 4.4, one gets

$$\begin{aligned} \partial^C \Delta_M(\overline{x}, \overline{y}) &\subset N^C(\mathrm{gph}\ M_r; \overline{x}, \overline{y}) \cap \big(l\mathbb{B}_{E^*} \times \Omega\big) \\ &= N^F(\mathrm{gph}\ M_r; \overline{x}, \overline{y}) \cap \big(l\mathbb{B}_{E^*} \times \Omega\big) \\ &\subset \partial^F \Delta_M(\overline{x}, \overline{y}). \end{aligned}$$

This ensures the Fréchet subdifferential regularity of Δ_M at $(\overline{x}, \overline{y})$, and the proof is finished. □

5. Applications to Metric Regularity of Set–Valued Mappings

Our purpose, in this section, is to give an application (to the metric regularity) of some results established in previous sections. We start with the well-known definition of metric regularity for set–valued mappings.

Definition 5.1. A set–valued mapping $M : E \Longrightarrow F$ is said to be *metrically regular* near $(\overline{x}, \overline{y})$ if there exist two real numbers $k \geq 0$ and $\delta > 0$ such that for all $(x, y) \in (\overline{x} + \delta\mathbb{B}_E) \times (\overline{y} + \delta\mathbb{B}_F)$ with $d\big(y, M(x)\big) \leq \delta$

$$d\big(x, M^{-1}(y)\big) \ \leq \ kd\big(y, M(x)\big).$$

It was shown in [**3**] that the metric regularity property of M is equivalent to the pseudo–Lipschitz property in the sense of Aubin [**1**] of its inverse set-valued mapping M^{-1} defined by $M^{-1}(y) := \{x \in E : y \in M(x)\}$. We refer for example to [**19, 21, 32**] and the references therein for the importance and several applications of this notion.

It has been shown (see for example [**19, 21**], that the condition

$$\text{(C.Q.)} \qquad \ker D^*_A M(\overline{x}, \overline{y}) \ = \ \{0\}$$

is sufficient for M to be metrically regular at $(\overline{x}, \overline{y})$, whenever M is assumed to be partially coderivatively compact (all set–valued mappings M whose graph is compactly epi–Lipchitz at $(\overline{x}, \overline{y})$ are partially coderivatively compact at $(\overline{x}, \overline{y})$. For more details about partial coderivative compactness property we refer the reader to [**21**]), at $(\overline{x}, \overline{y})$ and $D^*_A M(\overline{x}, \overline{y})$ denotes the coderivative of M at $(\overline{x}, \overline{y})$ associated with the Ioffe approximate normal cone (see for example [**18**] for the definition and properties of this cone).

In this section we will show that (C.Q.) is sufficient and necessary for M to be metrically regular at $(\overline{x}, \overline{y})$, under the Fréchet normal regularity of M at $(\overline{x}, \overline{y})$. To do this, we prove firstly the following proposition that gives a necessary condition for the lower-pseudo-Lipschitzness of the inverse set–valued mapping M^{-1} at $(\overline{y}, \overline{x})$ in terms of the Fréchet coderivative of M at $(\overline{x}, \overline{y})$. It is the key of the proof of our main result in this section. The proof of this proposition follows the same lines as in the proof of the necessity part in Theorem 5.8 in [**30**].

Proposition 5.2. *Let $M : E \Longrightarrow F$ be a set–valued mapping and let $(\overline{x}, \overline{y}) \in \text{gph } M$. If M^{-1} is lower–pseudo–Lipschitz at $(\overline{y}, \overline{x})$, then there exist $l \geq 0$ and $\alpha > 0$ such that for all $x \in (\overline{x} + \alpha\mathbb{B}_E) \cap M^{-1}(\overline{y})$, all $(x^*, y^*) \in E^* \times F^*$ with $x^* \in D^*_F M(x, \overline{y})(y^*)$ one has $\|y^*\| \leq l\|x^*\|$.*

Proof. Assume that M^{-1} is lower–pseudo-Lipschitz at $(\overline{y}, \overline{x})$. By Lemma 3.3, there exist $l \geq 1$ and $\alpha > 0$ such that

$$d\big(x, M^{-1}(y)\big) \ \leq \ l\|y - \overline{y}\|, \tag{5.1}$$

for all $x \in (\overline{x} + \alpha\mathbb{B}_E) \cap M^{-1}(\overline{y})$ and all $y \in (\overline{y} + \alpha\mathbb{B}_F) \cap \text{dom } M^{-1}$. Fix any $x \in (\overline{x} + \alpha\mathbb{B}_E) \cap M^{-1}(\overline{y})$ and any $(x^*, y^*) \in E^* \times F^*$ with $x^* \in D^*_F M(x, \overline{y})(y^*)$. Then $(x^*, -y^*) \in N^F(\text{gph } M; x, \overline{y})$. Fix now any $\varepsilon > 0$ and $\gamma > 1$. By the definition of the Fréchet normal cone, there exists $\delta \in]0, \alpha[$ such that for all $y \in \overline{y} + 2\delta\mathbb{B}_F$ and all $x' \in x + 2\delta\mathbb{B}_E$ with $(x', y) \in \text{gph } M$ one has

$$\langle x^*, x' - x\rangle - \langle y^*, y - \overline{y}\rangle \ \leq \ \varepsilon\|(y, x') - (\overline{y}, x)\|. \tag{5.2}$$

Now fix any $\widetilde{y} \in \overline{y} + l^{-1}\delta\mathbb{B}_F$. Then $\|\widetilde{y} - \overline{y}\| \leq l^{-1}\delta \leq l^{-1}\alpha \leq \alpha$ and so by (5.1) there exists $\widetilde{x} \in M^{-1}(\widetilde{y})$ such that $\|x - \tilde{x}\| \leq l\|\widetilde{y} - \overline{y}\| \leq ll^{-1}\delta = \delta \leq \gamma\delta$. By (5.2) we have

$$\langle y^*, \overline{y} - \widetilde{y}\rangle \leq \delta\gamma\|x^*\| + \epsilon(l^{-1}\delta + \delta\gamma) = \big(\gamma\|x^*\| + \varepsilon(l^{-1} + \gamma)\big)\delta,$$

and since this holds for any $\widetilde{y} \in \overline{y} + l^{-1}\delta\mathbb{B}_F$, one gets

$$l^{-1}\delta\|y^*\| \ \leq \ \big(\gamma\|x^*\| + \varepsilon(l^{-1}+\gamma)\big)\delta$$

and hence

$$l^{-1}\|y^*\| \ \leq \ \gamma\|x^*\| + \varepsilon(l^{-1}+\gamma).$$

By letting $\varepsilon \downarrow 0$ and $\gamma \downarrow 1$, one obtains

$$\|y^*\| \ \leq \ l\|x^*\|.$$

This completes the proof. □

Now, we can state our main result of this section.

Theorem 5.3. *Let E and F be two Asplund spaces. Suppose that $M : E \Longrightarrow F$ is partially coderivatively compact and Fréchet normally regular at $(\overline{x}, \overline{y})$. Then M is metrically regular at $(\overline{x}, \overline{y})$ if and only if (C.Q.) holds.*

Proof. *Sufficiency.* It is obvious, in view of the discussion after Definition 5.1, that (C.Q.) is sufficient.

Necessity. Suppose that M is metrically regular and Fréchet normally regular at $(\overline{x}, \overline{y})$. Then its inverse set–valued mapping M^{-1} is lower–pseudo–Lipschitz at $(\overline{y}, \overline{x})$. Fix now any $y^* \in \ker D^*_A M(\overline{x}, \overline{y})$, i.e., $0 \in D^*_A M(\overline{x}, \overline{y})(y^*)$. As the spaces E and F are Asplund and by the Fréchet normal regularity of M at $(\overline{x}, \overline{y})$ one has $D^*_A M(\overline{x}, \overline{y}) = D^*_F M(\overline{x}, \overline{y})$ and hence $0 \in D^*_F M(\overline{x}, \overline{y})(y^*)$. Therefore, Proposition 5.2 yields that $y^* = 0$, which ensures that $\ker D^*_A M(\overline{x}, \overline{y}) = \{0\}$. The proof then is complete. □

It is worth mentionning that the necessary and sufficient condition for metric regularity of set–valued mappings in the form of (C.Q.) was first obtained by Mordukhovich with his nonconvex generalized differentials (see [**29**] for finite dimension spaces and [**32**] for Asplund spaces).

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