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ON A MODEL FOR THE TERM STRUCTURE OF INTEREST RATE PROCESSES OF STABLE TYPE

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Abstract. Let $M(t), t \geq 0$, be a one-dimensional symmetric stable process of index $0 < \alpha \leq 2$. As a model for the term structure of interest rate processes we consider $r(t) = \mathcal{G}(t, M \circ T(t))$ where $\mathcal{G}$ and T are some functions. We show that this model includes, in particular, some models which can be described as solutions of Ito stochastic differential equations driven by the process M. We also construct a sequence of simple processes (random walks) which converge in distribution to the interest rate processes $r(t)$.

1. Introduction

As is well known, an interest rate process describes the profitability of a financial instrument, such as stock, bond or option. Hence if the price change is given by the sequence $X = (X_n)_{n \geq 0}$, then the interest rate process has in the simplest case the form

$$r_n = \frac{\triangle X_n}{X_{n-1}}, \text{ where } \triangle X_n = X_n - X_{n-1}.$$

The role of interest rate can better be understood by writing the sequence (X_n) in the form $X_n = X_0 \exp(H_n)$ with $H_n = \sum_{i=1}^{n} h_i$. It is easily verified that $X_n = X_0 \prod_{i=1}^{n}(1 + r_i)$, where $r_i = \ln(1 + h_i)$. In other words, if X_0 denotes the initial value of the financial sequence X, then the nth value is equal to the product of X_0 and n consecutive return-values.

Generating the above relation to continuous time, we obtain

$$dX_t = r_t X_t dt, \ t \geq 0.$$

In practice, however, another equivalent form of dependence between processes (r_t) and (X_t) is often used, namely the equation

$$dr_t = r_t dX_t$$

or, more generally,

$$dr_t = \sigma(t, r_t) dX_t, \ t \geq 0,$$

where $\sigma(t, x)$ is a given coefficient. The process (r_t) is called here according to the terminology of financial mathematics a *short interest rate process.*

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If the process X is a semimartingale, the equation above writes then

$$r_t = r_0 + \int_0^t \sigma(s, r_{s-}) dX_s,$$

where the integral on the right-hand side is understood as the stochastic integral with respect to a semimartingale and $r_{s-} = \lim_{t \uparrow s} r_t$ for all $s \geq 0$ (cf. [**13**], [**17**]). The interest rate processes of diffusion type

$$dr_t = a(t, r_t)dt + b(t, r_t)dW_t, \ t \geq 0 \tag{1.1}$$

play an important role in the case of continuous semimartingales. Here $a, b : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ are measurable functions and W is a one-dimensional Brownian motion with $W_0 = 0$.

W. M. Schmidt [**16**] considered a short interest rate model of the form

$$r_t = F\Big(f(t) + g(t)W^*_{T(t)}\Big), \tag{1.2}$$

where W^* is a Brownian motion, $T(t), F(x)$ are nonnegative, continuous, strictly increasing functions in $t \geq 0$ and $x \in \mathbb{R}$, respectively, and $f(t), g(t)$ are continuous functions with $g(t) > 0, \ t \geq 0$. The Schmidt's model (1.2) includes some known models of interest rate processes (r_t) being solutions of the stochastic equation (1.1) which can be obtained from (1.2) by an appropriate choice of "driving" components F, f, g and T. Moreover, some random walk models approximating the continuous model (1.2) were presented in [**16**].

The goal of this note is to consider the following generalization of the stochastic differential equation (1.1)

$$r_t = r_0 + \int_0^t a(s, r_s)ds + \int_0^t b(s, r_s)dM_s, \ t \geq 0, \tag{1.3}$$

as a model for interest rate processes (r_t). The process M in (1.3) is assumed to be a symmetric stable process of index $\alpha \in (0, 2]$ and the integral with respect to M is a stochastic integral of Ito type with respect to a stable process (cf. [**5**], [**15**]). The process (r_t) has the property of "heavy tails" distributions similar to the driving process M ([**7**]) that is verified by many real financial data for short rates ([**17**], Chap. 4).

Generalizing the Schmidt model (1.2), we introduce in section 2 the following model for (r_t)

$$r_t = F\Big(f(t) + g(t)M^*_{T(t)}\Big), \tag{1.4}$$

where M^* is a symmetric stable process of index α and the functions F, f, g and T satisfy the same assumptions as in model (1.2). Using properties of Ito integrals with respect to stable processes, we consider in section 3 some particular cases of equation (1.3) the solutions of which can be represented in the form (1.4).

Section 4 is devoted to construction of a sequence of simple interest rate processes $(r^n_t), n \geq 1$, which converges in law to the process (r_t). The processes r^n are presented as appropriate functionals of finite sums of i.i.d. random variables which belong to the domain of attraction of the corresponding symmetric stable process. The described approximation procedure which is of independent interest consists in the approximation of an arbitrary symmetric stable process through some more elementary stable random walk processes. We prove the convergence of the distributions of simple random walk processes on the space of their trajectories. This result generalizes the results of Gorenflo and Mainardi ([**9**], [**10**]) who considered random walk approximation procedures for stable processes and proved their convergence to the one-dimensional distributions of a stable process, i.e., for every fixed time-parameter $t > 0$.

2. A Model for Interest Rate Processes of Stable Type

Let $\mathbb{D}_{[0,\infty)}(\mathbb{R}) = \{x : [0,\infty) \to \mathbb{R}\}$ (or simply $\mathbb{D}$) be the space of right continuous functions with finite left limits which is often called the space of *cadlag functions*.

Definition 2.1. *A process $(M_t)_{t\geq 0}$ with $M_0 = 0$ and trajectories in $\mathbb{D}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is called a stable process of index $\alpha \in (0, 2]$ if it is stochastically continuous and*

$$\mathbf{E}\Big(e^{i\lambda(M_t - M_s)} \mid \mathcal{F}_s^M\Big) = \exp\Big((t-s)\Big[i\lambda\gamma + C_1 \int_{-\infty}^{0} (e^{i\lambda u} - 1 - \frac{i\lambda u}{1+u^2})\nu(du) +$$

$$C_2 \int_{0}^{\infty} (e^{i\lambda u} - 1 - \frac{i\lambda u}{1+u^2})\nu(du)\Big]\Big), \tag{2.1}$$

where $\mathcal{F}_s^M = \sigma(M_u : 0 \leq u \leq s)$ for all $0 \leq s \leq t < \infty$, γ, C_1 and C_2 are some real constants and

$$\nu(du) = \frac{du}{\mid u \mid^{1+\alpha}}.$$

The measure ν is called the *Levy measure* of the process M. The equality (2.1) can be also written in the form (cf. [**18**], Chap. 5)

$$\mathbf{E}\Big(\exp i\lambda(M_t - M_s) \mid \mathcal{F}_s^M\Big) = \exp\Big((t-s)[i\delta\lambda - c \mid \lambda \mid^{\alpha} (1 - iq\frac{\lambda}{\mid \lambda \mid} k(\lambda, \alpha)]\Big)$$

for all $t > s \geq 0,\ \lambda \in \mathbb{R}$, where

$$k(\lambda, \alpha) = \begin{cases} \tan \pi\alpha/2 & \text{if } \alpha \neq 1, \\ (2/\pi) \ln \mid \lambda \mid & \text{if } \alpha = 1. \end{cases}$$

Here $c > 0,\ \delta \in \mathbb{R},\ q \in [-1, 1]$ are the scale, location and skewness parameters, respectively (cf. [**18**]). The parameter α is called the stability index of the process M. In the special case of $\delta = 0$ and $q = 0$ (respectively, $C_1 = C_2$ and $\gamma = 0$) the process M is called a *symmetric stable process* of the index α. Using properties of stable distributions it can be verified that for all $\tau > 0$ the process $(M_{\tau t})$ is again a stable process with the same parameters α and q but with other parameters δ and c. We can always reduce the process M to another stable process with $c = 1$ which is called the standard process. In the symmetric standard case with $\alpha = 2$ the process M is a Brownian motion, which is the only stable process with continuous trajectories. For $\alpha < 2$ the trajectories of M are purely discontinuous. In the case of $1 < \alpha < 2$ the process is a martingale and for $0 < \alpha \leq 1$ it is a process of locally bounded variation [**1**].

When $\alpha \in (0,2]$, the process M is a semimartingale as a Levy process and the stochastic integral $\int_0^t f_s dM_s$ can be defined via the general semimartingale approach (cf. chapter 2 in [**14**]). From another side, the given stochastic integral can also be defined as Ito stochastic integral using some isometry properties between the corresponding spaces of integrators and integrals. The later construction is due to K.Ito who originally applied it to define the integral in the case of $\alpha = 2$ (cf. chapter 2 in [**12**]). For the case of an arbitrary $\alpha \in (0,2]$, the Ito's approach was generalized by J.Rosinski and W.Woyczynski [**15**].

The isometry property in the case of $\alpha = 2$ relies on the identity

$$\mathbf{E}[\int_0^t f_s dW_s]^2 = \mathbf{E}\int_0^t f_s^2 ds$$

for all predictable processes f_s for those the stochastic integral exists. For $\alpha \in (0,2)$, one has the following quasi-isometrical property: there exist constants c_α and C_α depending on α only such that for all $t > 0$

$$c_\alpha \mathbf{E}\int_0^t |f_s|^\alpha ds \le \sup_{\lambda>0} \lambda^\alpha \mathbf{P}\Big(\sup_{s\le t} |\int_0^s f_u dM_u| > \lambda\Big) \le C_\alpha \mathbf{E}\int_0^t |f_s|^\alpha ds.$$

One of the main advantages of the Ito approach to stochastic integrals with respect to symmetric stable processes is the following useful "inner clock" property that we shall use in this note as well.

Proposition 2.2. *(cf. Theorem 3.1 in* [**15**]*) Let H be an $\mathbb{F}^M$-adapted process*[1] *such that for all $t > 0$*

$$\mathbf{E}\int_0^t |H_s|^\alpha ds < \infty$$

and $\tau_t := \int_0^t |H_s|^\alpha ds \to \infty$ as $t \to \infty$. Let

$$\tau_t^{-1} = \inf\{s \ge 0 : \tau_s > t\}, \quad \bar{\mathcal{F}}_t = \mathcal{F}_{\tau_t^{-1}}.$$

Then, the process

$$\bar{M}_t = \int_0^{\tau_t^{-1}} H_s dM_s$$

is a $\bar{\mathbb{F}}$-adapted symmetric stable process of the same index α. Equivalentely,

$$\int_0^t H_s dM_s = \bar{M}_{\tau_t},$$

that is, the stochastic integral $\int_0^t H_s dM_s$ is nothing but a symmetric stable process of the same index with "inner clock" τ_t.

For all $0 < \alpha \le 2$, M is a Markov process and can be characterized in terms of analytic characteristics of Markov processes. First, for any function $f \in L^\infty(\Re)$ and $t \ge 0$, we can define the operator

$$(P_t f)(x) := \int_\Omega f(x + Z_t) d\mathbf{P}(\omega)$$

[1]That is, H_t is $\mathcal{F}_t^M$-measurable for all $t \ge 0$

where $L^\infty(\Re)$ is the Banach space of functions $f : \Re \to \Re$ with the norm $\|f\|_\infty = ess\sup|f(x)|$. The family $(P_t)_{t\geq 0}$ is called the family of convolution operators associated with M. Formally, for a suitable class of functions $g(x)$, let

$$(\mathcal{L}g)(x) = \lim_{t\downarrow 0} \frac{(P_t g)(x) - g(x)}{t}$$

called the infinitesimal generator of the process M.

It is known that for $\alpha < 2$

$$(\mathcal{L}g)(x) = \int\limits_{\Re\setminus\{0\}} [g(x+z) - g(x) - \mathbf{1}_{\{|z|<1\}} g'(x)z] \frac{k_\alpha}{|z|^{1+\alpha}} dz$$

for any $g \in C^2$, where C^2 is the set of all bounded and twice continuously differentiable functions $g : \Re \to \Re$ and k_α is a suitable constant. Formally, $\mathcal{L} = -(-\Delta)^{\alpha/2}$ where Δ is the Laplacian (second derivative operator) being the infinitesimal generator of a Brownian motion process ($\alpha = 2$). Therefore, in the case of $\alpha < 2$, the generator $\mathcal{L}$ is called sometimes *fractional derivatives operator*.

The Ito approach to stochastic integration with respect to stable (symmetric) processes shall be the main tool of study of our general model of interest rate processes. This model has the form

$$r_t = \mathcal{G}\Big(t, M^*_{T(t)}\Big),\ t \geq 0, \tag{2.2}$$

where the process M^* is a standard symmetric stable process of the index α, $T(t)$ with $T(0) = 0$ is a continuous strictly increasing function, and $\mathcal{G}(t,x)$ is a continuous in both variables function.

The model (1.4) is a special case of the equation (2.2). As in the Schmidt's model ($\alpha = 2$), this special case allows us to get more information about the process (r_t). It can easily be verified that the process $Y_t = f(t) + g(t)M^*_{T(t)}$ is again a stable process with the same stability index α and skewness parameter q, but with another location parameter δ and scale parameter c. If $S^t_{(\delta,c,q,\alpha)}(x), x \in \mathbb{R}$, is the distribution function of the stable variable M_t, therefore $S^t_{(0,1,0,\alpha)}$ is that of M^*_t, then the distribution function of r_t determined by (1.4) can be represented in the form $S^t_{(\delta,c,q,\alpha)}(x) \circ F^{-1}$, where F^{-1} is the inverse function to F.

3. Solutions of Stochastic Equations which can be Expressed by the General Model

A) Let us consider the following stochastic differential equation for the process (r_t)

$$dr_t = \Big(\theta(t) - \beta(t) r_t\Big)dt + \gamma(t) dM_t,\ t \geq 0 \tag{3.1}$$

which we call a *generalized Hull-White model* because in the case of $\alpha = 2$ it is known as Hull-White model ([**11**], [**19**]). We assume that the functions θ, β and γ are nonrandom.

Lemma 3.1. *The solution of equation (3.1) can be expressed in the form (1.4).*

Proof. We put

$$r_t = g(t)\Big[r_0 + \int\limits_0^t \frac{\theta(s)}{g(s)}ds + \int\limits_0^t \frac{\gamma(s)}{g(s)}dM_s\Big], \tag{3.2}$$

where M is any standard symmetric stable process of the index α, $r_0 \in \mathbb{R}$ is an arbitrary initial value and $g(t) = \exp\Big(-\int\limits_0^t \beta(s)ds\Big)$.

Using integration by parts formula for semimartingales ([**13**], p. 99) one can easily verify that the process (3.2) is a solution of the equation (3.1).

Let

$$T_t = \int\limits_0^t \mid \frac{\gamma(s)}{g(s)} \mid^\alpha ds,\ t \geq 0.$$

According to Proposition 2.2, the function (T_t) is an "inner clock" for the stochastic integral in (3.2). Moreover, if

$$\mid \frac{\gamma}{g} \mid^\alpha \in L^{loc} \text{ and } T_\infty = \lim_{t\to\infty} T(t) = \infty, \tag{3.3}$$

then there exists a stable process M^* of the same index α so that

$$\int\limits_0^t \frac{\gamma(s)}{g(s)}dM_s = M^*_{T(t)}$$

(cf. [**5**], [**15**]).

It follows then from (3.2) that

$$r_t = f(t) + g(t)M^\star_{T(t)},$$

where

$$f(t) = g(t)\Big[r_0 + \int\limits_0^t \frac{\theta(s)}{g(s)}ds\Big].$$

□

Remark 3.2. *In (3.3) the condition $T_\infty = \infty$ is not necessary. If $T_\infty < \infty$, then the stochastic integral in (3.2) can be represented as another symmetric stable process stopped at T_∞ (cf.* [**5**]*).*

B) Generalizing the case of $\alpha = 2$, we consider the stochastic equation

$$dr_t = r_t\Big(\theta(t) - \beta(t)\ln r_t\Big)dt + \gamma(t)r_t dM_t,\ t \geq 0, \tag{3.4}$$

which we call a *generalized Black-Karasinski model* (cf. e.g. [**2**]).

Lemma 3.3. *The model (3.4) can be reduced to the form (1.4).*

Proof. Obviously, we can rewrite the equation (3.4) in the equivalent form

$$d\ln r_t = \Big[\theta(t) - \beta(t)\ln r_t\Big]dt + \gamma(t)dM_t,$$

which can be solved similarly as the equation (3.1). Its solution has the form

$$r_t = F\Big(f(t) + g(t)M^*_{T(t)}\Big),$$

where M^* is a symmetric stable process of the same index, the functions $g(t)$ and $T(t)$ are defined as in Lemma 3.1, and

$$f(t) = g(t)\Big[r_0 + \int_0^t \frac{\theta(s)}{g(s)} ds\Big],$$
$$F(x) = \exp(x).$$

□

C) In a similar way we can treat the following generalization of the model (3.1)

$$dr_t = \Big[\theta(t) - r_t\Big]dt + \gamma(t)dM_t^1,$$

where the coefficients $\theta(t)$ and $\gamma(t)$ are the solutions of stochastic equations

$$d\theta(t) = \Big[\theta - \theta(t)\Big]dt + \beta_1(t)dM_t^2,$$

and

$$d\gamma(t) = \Big[\gamma - \gamma(t)\Big]dt + \beta_2(t)dM_t^3,$$

respectively, and M^1, M^2, M^3 are independent symmetric stable processes of the same index (cf. [**3**], and [**17**], Chap. 3, sec. 4).

D) We call the model given by

$$dr_t = \theta(t)r_t dt + \gamma(t)r_t dM_t$$

a *generalized Dothan model* (see [**4**]) where M is a symmetric stable process of index $\alpha \in (0, 2]$. Obviously, the solution (r_t) has the form (1.4) with

$$F(x) = \exp(x), \quad f(t) = \int_0^t \theta(s)ds,$$

$$g(t) \equiv 1, \quad T(t) = \int_0^t \mid \gamma(s) \mid^p ds.$$

Remark 3.4. *The generalized Dothan model can be seen as a model for geometric stable motion generalizing the Dothan model as a model for geometric Brownian motion.*

4. The Convergence of Distributions of Random Walk Processes to the Distributions of Stable Processes

In theory of random processes as well as in its many applications it is important to find the distributions of various functionals of some basic stochastic processes. Thus, the formula (1.4) or the general formula (2.2) are examples of such functionals of symmetric stable processes. It is also important to know the distributions of the interest rate processes in the corresponding financial problems. For example, so-called contingent claims $C(r_t)_{0 \le t \le \tau}$ on the bond markets depending on (r_t) are usually expressed in the form

$$C(r_t) = \mathbf{E_Q}\Big(\exp\{-\int_t^\tau r_u du\}C(r_\tau) \mid \mathcal{F}_t\Big),$$

where τ is the bond maturity time, $(\mathcal{F}_t)$ is a filtration, Q is an appropriate martingale measure, and C is a known deterministic function (cf. [**16**]). In order to calculate $C(r_t)$, one needs to know the distribution of the process (r_t). In particular, for the pricing of contingent claims it would be usefull to have a sequence of more simple processes (r_t^n), $n \geq 1$, which converges in a suitable sense to the process (r_t). It would allow us to use the corresponding sequence

$$\mathbf{E_Q}\Big(\exp\{-\int_t^\tau r_u^n du\}C(r_\tau^n) \mid \mathcal{F}_t\Big),\ n \geq 1,$$

to approximate the claim $C(r_t)$.

Here we shall construct a sequence of processes (r_t^n) which approximates the process (r_t) of the form (2.2). For every fixed n, the process (r_t^n) will be determined as a sum consisting of n independent and identically distributed random variables. Furthermore, we shall show that $r^n \to^{\mathcal{D}} r$ as $n \to \infty$ meaning the convergence of distributions of processes r^n to the distribution of the process r.

First, we note that the characteristic function (2.1) of the symmetric stable process can be written in the form

$$\mathbf{E}e^{i\lambda M_t} = \exp\Big\{t\int_{-\infty}^{\infty}(e^{i\lambda u} - 1 - \frac{i\lambda u}{1+u^2})\frac{1+u^2}{u^2}dG(u)\Big\},$$

where

$$dG(u) = \frac{u^2}{1+u^2}\nu(du).$$

Obviously, $G(u)$ is a monotone and bounded function since the density of the measure dG with respect to the Lebesgue measure is integrable over $\Re$.

We recall also that a random variable ξ has a symmetric stable distribution of the index $\alpha \in (0,2]$ if its characteristic function has the following form

$$\psi_\alpha(\lambda) = \mathbf{E}e^{i\lambda\xi} = \exp\Big(\int_{-\infty}^{\infty}(e^{i\lambda u} - 1 - \frac{i\lambda u}{1+u^2})\nu(du)\Big).$$

The function

$$U_\alpha(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda x}\psi_\alpha(\lambda)d\lambda,\ x \in \mathbb{R},$$

coincides then with the distribution function of ξ.

Let $\Big(\xi_n\Big)_{n\geq 1}$ be a sequence of i.i.d. symmetric random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. For every $n \in \mathbb{N}$ and $\tau \in [0,\infty)$, we consider the stochastic process $(M_{t,\tau}^n), t \in [0,\tau]$, defined as

$$M_{t,\tau}^n = \begin{cases} \sum_{i=1}^k \xi_i \text{ if } t \in [\frac{k-1}{n}\tau, \frac{k}{n}\tau), \\ \sum_{i=1}^n \xi_i \text{ if } t = \tau. \end{cases} \tag{4.1}$$

We call $M_{\cdot,\tau}^n$ a *random walk process.* For every $n \in \mathbb{N}$ and $\tau \in [0,\infty)$, $M_{\cdot,\tau}^n$ is a cadlag process with independent increments. In particular, $M_{\cdot,\tau}^n \in \mathbb{D}_{[0,\tau]}(\mathbb{R})$.

We are interested in conditions guaranteeing the $\mathcal{D}$- convergence of processes $(M_{t,\tau}^n)_{0\leq t\leq\tau}$ to a symmetric stable process $(M_t)_{0\leq t\leq\tau}$.

Set $S_n := \sum_{i=1}^n \xi_i,\ n \geq 1$, and let F be the common distribution function of random variables $\xi_i, i \geq 1$.

Definition 4.1. *We say that the distribution function F belongs to the (normal) domain of attraction of the distribution U which is not concentrated in one point if there are constants $a_n > 0$ such that, for all $x \in \mathbb{R}$, $P_{\frac{1}{a_n}S_n}(x) \to U(x)$ as $n \to \infty$ where $P_{\frac{1}{a_n}S_n}$ denotes the distribution function of $\frac{1}{a_n}S_n$.*

It is known that only the distributions $U_\alpha(x), x \in \mathbb{R}, \alpha \in (0,2]$, have a domain of attraction.

Define

$$\gamma_n = n \int_{-\infty}^{\infty} \frac{x}{1+x^2} dF_n(x),$$

$$G_n(x) = n \int_{-\infty}^{x} \frac{u^2}{1+u^2} dF_n(u),$$

where F_n denotes the distribution function of random variables $a_n^{-1}\xi_i, i \geq 1$.

Definition 4.2. *For monotone and bounded functions $G_n(x)$, $G(x)$ we write $G_n(x) \Rightarrow G(x)$ as $n \to \infty$ to denote the convergence in all points of continuity $x \in \mathbb{R}$.*

The next result is due to A. V. Skorohod and gives a sufficient condition for the $\mathcal{D}$-convergence of sums of i.i.d. random variables to a symmetric stable process.

Proposition 4.3. *(cf. Theorems 1, 2 in* [**8**]*, Chap. 9, §6). If $\gamma_n \to 0$ and $G_n(x) \Rightarrow G(x)$ as $n \to \infty$, then, for every $\tau \in [0,\infty)$, it holds*

$$(M^n_{t,\tau})_{0\leq t\leq\tau} \to^{\mathcal{D}} (M_t)_{0\leq t\leq\tau} \quad as \quad n \to \infty.$$

Now we consider the following sequences of numbers $\Big(p_k(\alpha)\Big)_{k\in\mathbb{Z}}$ defined as

$$\begin{cases} p_0 = 1-\mu, \\ p_{\mp k} = (-1)^{k+1}\mu d\binom{\alpha}{k},\ k = 1,2,... \end{cases}$$

for $0 < \alpha < 1$ and $0 < \mu \leq \cos(\frac{\alpha\pi}{2})$, or

$$\begin{cases} p_0 = 1+\mu\alpha(\cos\frac{\alpha\pi}{2})^{-1}, \\ p_{\mp 1} = -\mu(d\binom{\alpha}{2}+d), \\ p_{\mp k} = (-1)^k\mu d\binom{\alpha}{k+1}, \text{k=2, 3,...} \end{cases}$$

for $1 < \alpha \leq 2$ and $0 < \mu \leq \alpha^{-1}\cos(\frac{\alpha\pi}{2})$, or

$$\begin{cases} p_0 = 1-\frac{2\mu}{\pi}, \\ p_{\mp k} = \mu\pi \mid k \mid (\mid k \mid +1), \text{k=1, 2,...} \end{cases}$$

for $\alpha = 1$ and $0 < \mu \leq \frac{\pi}{2}$, where $d = \frac{\sin\frac{\pi}{2}\alpha}{\sin\pi\alpha}$.

It can easily be seen that $p_k(\alpha) \geq 0,\ k \in \mathbb{Z}$, and $\sum_{k\in\mathbb{Z}} p_k(\alpha) = 1$ for all $\alpha \in (0,2]$ (cf. [**10**]), i.e., $\Big(p_k(\alpha)\Big)_{k\in\mathbb{Z}}$ defines a probability distribution. We note that it is also symmetric.

We assume that

$$P(\xi_i = k) = p_k(\alpha),\ k \in \mathbb{Z} \tag{4.2}$$

for all $i \geq 1$. The next statement is an immediate consequence of Theorem 3.2 in [**10**].

Proposition 4.4. *For all $\alpha \in (0, 2]$, the probability distribution function F_α defined as*

$$F_\alpha(x) = \sum_{k \leq x} p_k(\alpha),\ x \in \mathbb{R},$$

belongs to the domain of attraction of the distribution U_α.

Theorem 4.5. *Let the sequence $(M^n_{t,\tau}), t \in [0, \tau]$, is defined as in (4.1) and the corresponding i.i.d. symmetric random variables $\xi_i, i \geq 1$, have the distribution $\Big(p_k(\alpha)\Big)_{k\in\mathbb{Z}}$. Then, it holds*

$$(M^n_{t,\tau})_{0\leq t\leq\tau} \to^{\mathcal{D}} (M_t)_{0\leq t\leq\tau} \text{ as } n \to \infty$$

where M is a symmetric stable process of the index $\alpha \in (0, 2]$.

Proof. Acccording to Proposition 4.3, it suffices to show that

$$\gamma_n \to 0 \qquad \text{and} \qquad G_n(x) \Rightarrow G(x),\ n \to \infty.$$

It follows from Proposition 4.4 that there exist constants $a_n > 0$ such that the sequence of distributions of random variables $a_n^{-1}\sum_{i=1}^n \xi_i$ converges to the distribution U_α as $n \to \infty$ where the random variables ξ_i are defined in (4.2). In order to check the conditions of the Proposition 4.3, we remark that $F_n(x) = F_{n,\alpha}(x) := F_\alpha(a_n x)$.

Let ϕ_α be the characteristic function of the distribution F_α. Obviously, the statement F_α *belongs to the domain of attraction of* U_α is equivalent to

$$\phi^n_\alpha(\lambda a_n^{-1}) \to \psi_\alpha(\lambda) \text{ as } n \to \infty \tag{4.3}$$

for every $\lambda \in \mathbb{R}$.

It is easy to prove that the relation (4.3) can be written in the form

$$n\Big[\phi_\alpha(\lambda a_n^{-1}) - 1\Big] \to \omega(\lambda), \lambda \in \mathbb{R} \text{ as } n \to \infty, \tag{4.4}$$

where $\psi_\alpha(\lambda) = e^{\omega(\lambda)}$ (cf. Theorem 1 in [**6**], Chap. XVII, §1).

Following the steps as in the proof of the Theorem about the canonical representation of infinitely divisible characteristic functions ([**6**], Theorem 1, Chap. XVII, §2), we conclude from (4.4) that

$$nF_{n,\alpha}(dx) \Rightarrow \nu(dx) \text{ as } n \to \infty, \tag{4.5}$$

where the measure ν is defined in (2.1).

Taking into account the definition of the functions $G_n(x)$ and $G(x)$, we obtain

$$G_n(x) \Rightarrow G(x) \text{ as } n \to \infty.$$

The relation (4.5) also implies $\gamma_n \to 0$ as $n \to \infty$ what finishes the proof of the theorem. □

Theorem 4.5 can be extended to processes (r_t) of the form (1.4) or (2.2) in the following way. First we recall a well-known fact that, for every $\tau \in (0, \infty)$, the

Skorohod space $\mathbb{D}_{[0,\tau]}(\mathbb{R})$ is a separable metric space with respect to the metric ρ_D defined as

$$\rho_D(x,y) = \inf_{\lambda \in \Lambda}\{\sup \mid x(t) - y(\lambda(t)) \mid + \sup_t \mid t - \lambda(t) \mid\}$$

where $x, y \in \mathbb{D}_{[0,\tau]}(\mathbb{R})$ and $\Lambda = \{\lambda : [0,\tau] \to [0,\tau],\ \lambda$ are monotone and continuous functions with $\lambda(0) = 0, \lambda(\tau) = \tau\}$.

Corollary 4.6. *Set $r_t^n := \mathcal{G}(t, M_{T_t}^n)$ where the random walk process M^n is defined in (4.1), $\mathcal{G}(t,x)$ is a continuous in both variables function and $T(t)$ is a continuous, increasing function with $T_0 = 0$. Then, for every $\tau \in (0,\infty)$, it holds*

$$(r_t^n)_{0\le t\le\tau} \to^{\mathcal{D}} (r_t)_{0\le t\le\tau} \quad \text{as} \quad n \to \infty$$

where r is defined in (2.2).

The proof of Collorary 4.6 follows from Theorem 4.5 using the fact that $\mathcal{G}(t, [\cdot]_{T_t})$ is a continuous functional in the metric ρ_D on the space $\mathbb{D}_{[0,\tau]}(\mathbb{R})$ for every $\tau \in (0,\infty)$.

Remark 4.7. *We notice that the functional in (1.4) is a special case of the more general functional $\mathcal{G}(t, [\cdot]_{T_t})$ and, therefore, Collorary 4.6 remains also true for the model (1.4).*

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